Markov chains, diffusion and Green functions. Applications to traffic routing in *ad hoc* networks and to image restoration

Chaînes de Markov, diffusion et fonctions de Green. Applications au routage dans les réseaux ad hoc et à la restauration d'images

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Abstract

In this document, we recall overall main definitions and basic properties of Markov chains on finite state spaces, Green functions, partial differential equations (PDE's) and their (approximate) resolution using diffusion walks in a discrete graph. We apply then all these topics to the study of traffic propagation and repartition in *ad hoc* networks. Last we also apply this framework to image restoration (with and without boundaries.)

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Résumé

Nous rappelons dans un premier temps les définitions et propriétés fondamentales des chaînes de Markov à temps discret et à nombre d'états fini. Ce schéma permet la résolution (approchée) de certaines classes d'équations aux dérivées partielles par fonctions de Green discrètes. Il en résulte naturellement que l'on peut utiliser certaines classes de marches aléatoires dans des graphes finis pour résoudre ces problèmes. Nous montrons enfin l'application de l'ensemble de cette théorie à l'étude de la propagation du trafic dans des réseaux *ad hoc* réguliers et à la restauration d'images bruitées avec ou sans processus bords.

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1 Introduction

In this document, we present and recall several basic definitions and properties of:

- Markov chains on finite state spaces.
- Green functions.
- partial differential equations (PDE's) and their (approximate) resolution using diffusion walks in a discrete graph. In particular the role of boundary and source conditions modeling has been emphasized.
- application of these to traffic propagation and repartition in *ad hoc* networks, and to image restoration with eventual boundary process.

We try to present these topics in a pedagogical way, hoping this aim will be achieved.

2 Markov chains on a finite state space - Recalls

To start with we consider a **discrete finite** state space $\mathcal{E}$ with cardinal N .
Then we consider $E = \mathbb{R}^{|\mathcal{E}|}$ as the vector space of real **bounded functions** on $\mathcal{E}$:

$$\begin{aligned} \mathcal{E} &\mapsto \mathbb{R} \\ x &\mapsto f(x) \ . \end{aligned}$$

Two functions (or classes of functions) play a particular role in E:

- the family of indicatrix functions:

  $$e_{x_0} = \mathbb{1}_{x=x_0} \quad \forall x_0 \in \mathcal{E}$$

  which are identified to the canonical base vectors of E .
- the constant function $\mathbf{1}$ defined by:

  $$x \mapsto 1 \qquad \forall x \in \mathcal{E}$$

Now a probability distribution P is a positive *measure* on $\mathcal{E}$ *i.e.,* a (positive) **linear form** with total mass 1 on E. Defining: $Pr\ (X = x_0) = P(e_{x_0}) = P(\mathbb{1}_{x=x_0})\ \ \forall x_0 \in \mathcal{E}$ one has:

$$\begin{aligned} P(f) \ &= \sum_{x\in\mathcal{E}} P(e_x)\ f(x) \ = \sum_{x\in\mathcal{E}} Pr\ (X = x)\ f(x) \\ &= \mathbb{E}\,[\,f\,] \\ \text{with} \quad \sum_{x\in\mathcal{E}} Pr\ (X = x) \ &= 1 \ \text{ of course} \end{aligned}$$

This last condition can be written as

$$P(\mathbf{1}) = 1$$

Also useful in the sequel will be the uniform measure on $\mathcal{E}$ defined by:

$$\mu_0(e_x) = \frac{1}{|\mathcal{E}|} = \frac{1}{N} \quad \forall x \in \mathcal{E} \ .$$

2.1 stochastic matrices

definition
A stochastic matrix Q is such that

$$Q\ \mathbf{1} = \mathbf{1}$$

right-wise multiplication of a probability measure by a stochastic matrix
Let P be a probability measure and Q a (stochastic) matrix in $\mathcal{L}(E)$. Then by definition:

$$(P\ Q)(f) = P(Qf) \qquad \forall f \in E$$

In particular one has:

$$(P\ Q)(\mathbf{1}) = P(\mathbf{1}) = \mathbf{1}$$

and $P\ Q$ is thus definitely a probability measure on $\mathcal{E}$.
Now, let $y \in \mathcal{E}$. Then, the probability measure $Qr\ = P\ Q$ applied to $f(x) = \mathbb{1}_{x=y}$ writes:

$$\begin{aligned} Qr\ (\mathbb{1}_{x=y}) = (P\ Q)(e_y) &= P(Q(e_y)) \\ &= \sum_{x\in\mathcal{E}} Pr\ (X = x)\ Q_{x\,y} \end{aligned}$$

2.2 Markov chains on $\mathcal{E}$

Let us consider a Markov chain with initial associated probability measure P_0 and transition matrix Q. We have that:

$$Pr\ (X_0 = x_0, \ldots .....X_M = y) = Pr\ (X_0 = x_0)\ Q_{x_0\,x_1} \ldots Q_{x_{M-1}\,y} \tag{1}$$

so that by noting $P_M = Pr\ (X_M = .)$ one has very formally and concisely :

$$P_M = P_0\ Q^M$$

In particular:

$$P_M(y) = Pr\ (X_M = y) = \sum_{x_0\in\mathcal{E}} Pr\ (X_0 = x_0)\ (Q^M)_{x_0\,y}$$

and thus

$$Pr\ (X_M = y \mid X_0 = x_0) = (Q^M)_{x_0\,y}$$

A useful formula for the sequel is the following: let us compute
the expectation of some function $\psi \in E$ **at step** M (wrt. to previous Markov chain) .
Then:

$$\begin{aligned} \mathbb{E}\,[\,\psi(X_M)\,] = P_M(\psi) = (P_0\ Q^M)(\psi) &= P_0(Q^M\psi) \\ &= \sum_{x_0\in\mathcal{E}} Pr\ (X_0 = x_0)\ \ (Q^M\ \psi)(x_0) \end{aligned} \tag{2}$$

Results directly from previous formulaes. In particular:

$$\forall \psi \in E,\ \ \mathbb{E}\,[\,\psi(X_M) \mid X_0 = x_0\,] = (Q^M\ \psi)(x_0) \tag{3}$$

2.3 invariant measure - convergence

Recall that E is endowed with the L^∞ norm:

$$|| f ||_\infty = \sup_{x \in \mathcal{E}} |f(x)| \ .$$

so that the variation distance between two measures is defined equivalently as:

$$|| \mu - \nu ||_V = \sup_{|| f ||_\infty = 1} |\mu(f) - \nu(f)| \tag{4a}$$

$$= \sum_{x \in \mathcal{E}} |\mu(e_x) - \nu(e_x)| = 2 - \sum_{x \in \mathcal{E}} \min(\mu(e_x),\ \nu(e_x)) \tag{4b}$$

and satisfies of course from the first previous equality (4a) :

$$|\mu(f) - \nu(f)| \leq || \mu - \nu ||_V \ || f ||_\infty \quad \forall \mu,\ \nu,\ f$$

Now, the Dobrushin contraction coefficient of a **stochastic matrix** A is defined as:

$$c(A) = \frac{1}{2} \sup_{i,j} || A_i(.) - A_j(.) ||_V \quad , \quad 0 \leq c(A) \leq 1$$

It ensures that Winkler (1995) :

$$|| (\nu A) - (\mu A) ||_V \leq c(A) \ || \nu - \mu ||_V \quad \forall \mu, \nu, A$$
$$c(A\ B) \leq c(A)\ .\ c(B) \quad \forall A, B$$

In the case when A is **strictly positive**, one has from the second equality (4b) :

$$0 \leq c(A) < 1 \quad \text{(hence the contraction property)}$$

In this case, Perron-Frobenius theorem ensures a **unique** invariant measure μ of A *i.e.,* :

$$(\mu A) = \mu$$

Thus one has:

$$\forall M \geq 0, \quad || (P_0\ A^M) - \mu ||_V = || (P_0\ A^M) - (\mu A^M) ||_V$$
$$\leq c(A)^M \ || P_0 - \mu ||_V \quad \forall \text{ initial } P_0 \tag{5}$$

$\Rightarrow$ the measure $P_0\ A^M$ converges to μ as $M \to +\infty$, whatever initial measure P_0 .
This can be generalized when some power of A, A^r is strictly positive ($r > 1$), although A itself may be not so:
Indeed let us decompose in an euclidian manner: $M = nr + p,\ 0 \leq p < r$:

$$|| P_0\ A^{nr+p} - \mu_r\ A^p ||_V = || P_0\ (A^r)^n\ A^p - \mu_r\ (A^r)^n\ A^p ||_V$$
$$\leq c(A^r)^n \ || P_0\ A^p - \mu_r\ A^p ||_V \qquad 0 \leq p < r$$
$$\leq c(A^r)^n\ c(A)^p \ || P_0\ - \mu_r ||_V \qquad 0 \leq p < r \tag{6}$$

3 Discretization of a class of linear (elliptic) PDEs

We follow Ycart (1997); Rubinstein (1981) with slight changes. Find $f \in E$ such that:

$$\boxed{\begin{cases} \gamma\ (A - \mathrm{I})\ f = \phi\ , & A \in \mathcal{L}(E) \\ (\textit{eventually})\ \textit{subject to} & \\ f(x_s) = b(x_s)\ \text{known} & \forall x_s \in \mathcal{B} \subset \mathcal{E} \end{cases}} \tag{7}$$

Here γ is a constant, and $\mathcal{B}$ represents the boundary of domain $\mathcal{E}$ wiewed as a multi-dimensional (in general 2D) discrete lattice.

3.1 a linear algebra point of view

We split $\mathcal{E}$ in boundaries/non-boundaries, so that the following decomposition holds:

$$\begin{aligned} \mathcal{E} &= \tilde{\mathcal{E}} \cup \mathcal{B} \quad \text{(state space)} \\ E &= \tilde{E} \oplus B \quad \text{(functional vector space)} \\ f &= \tilde{f} + b \quad \text{(functions)} \end{aligned}$$

Problem (7) writes thus: $\gamma\ (A - \mathrm{I})\ \tilde{f} = \phi + \gamma\ (I - A)\ b \quad with \quad A \in \mathcal{L}(E)$
Now let us note $\tilde{P}$ the linear projector on susbspace $\tilde{E}$ with kernel $\mathcal{B}$. One has:
$\tilde{f} = \tilde{P}f = \tilde{P}\tilde{f}$ and $\tilde{P}\ b = 0$. Left-application of $\tilde{P}$ to previous equation yields:

$$\boxed{\gamma\ (\tilde{A} - \mathrm{I})\ \tilde{f} = \psi = \tilde{P}\phi - \gamma\ \tilde{P}A\ b \quad with \quad \tilde{A} = \tilde{P}A \in \mathcal{L}(\tilde{E})} \tag{8}$$

Note that $\tilde{P}A\tilde{P} = \tilde{A}\tilde{P}$ and $\tilde{P}\phi$ are the restrictions of A (resp. ϕ) to $\tilde{E}$.
In the sequel A will often be a **stochastic** matrix (see below the Poisson-Laplace case) so that $\mathbf{1}$ is an eigenfunction of A with eigenvalue $\lambda_1 = 1$. In many cases the multiplicity of λ_1 is 1 (see below) and all other eigenvectors have eigenvalues such that $|\lambda_i| < 1$.
Thus solving (8) with proper invertibility conditions yields:

$$\tilde{f} = -\frac{1}{\gamma}\ (\mathrm{I} - \tilde{A})^{-1}\ \psi \tag{9}$$

3.1.1 an example: the Laplace and Poisson equations

$\mathcal{E}$ is now a discrete sublattice of $\mathbb{Z}^2$ with lattice step h (and thus unit cell size: $h \times h$.)
This sublattice is endowed with a non-oriented, **connected** graph structure, the neigborhood relationship being noted as $\mathbf{x_t} \sim x_s$ (*e.g.* 4-connectivity .) Now:

$$\begin{aligned} \Delta f(x_s) &\approx \frac{[\ \sum_{x_t \sim x_s} f(x_t)\ ] - 4\ f(x_s)}{h^2} \Rightarrow \Delta f \approx \gamma\ (A - I)\ f \\ \text{with } \gamma = \frac{4}{h^2} \quad \text{and } (Af)(x_s) &= \frac{1}{4}\ [\ \sum_{x_t \sim x_s} f(x_t)\ ] \quad \forall x_s \in \mathcal{E}\ (\textbf{averaging operator}) \\ \text{In this case: } B(x_s) &= \gamma\ \tilde{P}A\ b(x_s) = \sum_{\substack{x_t \sim x_s \\ x_t \in \mathcal{B}}} b(x_t) \qquad \forall x_s \in \tilde{\mathcal{E}}\ . \end{aligned} \tag{10}$$

It is of paramount importance to note here that A is a **stochastic** matrix on $\mathcal{E}$.
A result of Mohar (1997) states that the multiplicity of $\lambda_1 = 1$ is 1 for **connected** graphs.
Also the Perron-Frobenius theorem implies that all other eigenvectors have corresponding eigenvalues $|\lambda_i| < 1$.

Now the corresponding solution of (7) for $\phi(x) = \frac{\mathbb{1}_{x=y}}{h^2} = \frac{e_y}{h^2}$ is noted as [1]

$$G_{x\,y} \quad \forall x, y \in \tilde{\mathcal{E}} \quad (\text{“discrete” Green function}) \ .$$

From (9) its closed form for null boundary conditions $b = 0$ writes as:

$$G_{x\,y} = -\frac{1}{4}\left((\mathrm{I} - \tilde{A})^{-1} e_y \right)(x) = -\frac{1}{4}\left((\mathrm{I} - \tilde{A})^{-1} \right)_{x\,y} \tag{11}$$

By previous argument ($|\lambda_i| < 1$) this expression can be expanded in series as

$$G_{x\,y} = -\frac{1}{4}\left(\sum_{M \geq 0} (\tilde{A}^{\,M})_{x\,y} \right) \qquad x,\ y \in \tilde{\mathcal{E}} \tag{12}$$

We have thus solved the discrete analogue of the continuous Poisson problem:

find $f(x) = G(x, y)$ such that:

$\Delta_x G(x, y) = \delta(x, y)$ and $G(x, y) = 0 \ \ \forall x \in \partial\mathcal{D}$ (boundary conditions)

The conditions of convergence of the series (12) are examined in the following section .

3.2 interpretation in terms of Markov Random Fields (MRFs)

We follow Mohar (1997) [2] .
We first endow E (and thus $\tilde{E}$), with the usual ℓ^2-like, scalar product:

$$< f, g >= \sum_{x_s \in \mathcal{E}} f(x_s)\ g(x_s)$$

Now the basic energy (quadratic form) associated to the Laplacian operator is:

$$U(f) = \sum_{x_s \sim x_t} a_{s,t}\ \left(f(x_s) - f(x_t)\right)^2$$

where the **positive** weights $a_{s,t}$ are noted with a comma to emphasize that they are **symmetric** (most often they are equal to 1 in our case of interest: *e.g.* Laplace-Poisson).
This corresponds of course to a **Gaussian** Markov Random Field.
Notice also that the $a_{s,t}$ can be interpreted as a **contour** process in image processing.

[1] we divide unit function e_y by unit cell size h^2 since it yields a “discrete approximation” of the δ distribution. This can be seen for instance by the fact that: $\sum_{y \in \mathcal{E}} \frac{e_y}{h^2}\ h^2 = 1$. Notice also that h disappears in following discrete linear equations due to the particular case of a 2nd degree PDE in 2D.

[2] notice that here we note $x_s, x_t \ldots$ the topological pixel **sites** instead of their values as usually done with MRFs.

3.2.1 Dirichlet boundary conditions

Let us minimize $U(f)$ subject to $f(x_s) = b(x_s) \quad \forall x_s \in \mathcal{B}$.

This corresponds to solve problem (8) with $\phi = 0$ and we have thus:

$$\left\{ \begin{array}{lll} (1 - \tilde{A})\ \tilde{f} & = \tilde{P}\ A\ b & \textit{and} \\ \tilde{A}_{st} & = \dfrac{a_{s,t}}{\sum\limits_{x_t \sim x_s} a_{s,t}} & \textit{is a Laplacian transfer matrix.} \end{array} \right.$$

This is a Laplace-like problem with Dirichlet boundary conditions, which correspond to **absorption** (see subsection 3.3: Markov chains interpretation).
The matrix $\tilde{A}$ is thus **symmetric** and **semi-stochastic** :
$\sum\limits_t |\tilde{A}_{st}| < 1$ when $\exists x_t \sim x_s$ *s.t.* $x_t \in \mathcal{B}$ *i.e.,* near boundaries.
For this case, Magnus (2007) uses:

Theorem (Hadamard) 1 *: A square matrix C verifying $C_{ii} > \sum\limits_{j \neq i} |C_{ij}| \quad \forall i = 1 \ldots N$ (strongly dominant [by lines]) is* **non-singular**,

and namely, the following subsequent refinement:

Theorem (Magnus) 2 *: A square real matrix C verifying:*

(1) $C_{ii} > 0\ ;\ C_{ij} \leq 0 \quad \forall i, j = 1 \ldots N, j \neq i$ *;*

(2) $C_{ii} \geq \sum\limits_{j \neq i} |C_{ij}| \quad \forall i = 1 \ldots N$ *(dominance)* ,

(3) $C_{ii} > \sum\limits_{j \neq i} |C_{ij}|$ *for at least one* i ,

(4) C is undecomposable i.e., *precisely the matrix C has no strictly smaller stable square submatrix* ,

is **invertible** *and its inverse is a* **positive** *matrix.*

by applying this Theorem to $C = \mathrm{I} - \tilde{A}$, it is enough that one of the sum values satisfy $\sum\limits_j |\tilde{A}_{ij}| < 1$ to ensure that $\mathrm{I} - \tilde{A}$ is invertible, ensuring thus (from Perron-Frobenius) that the series $(\tilde{A}^{\,M})$ converges.

Remark it is important to see that the non-decomposability condition (4) is necessary : otherwise one could construct C as

$$C = \left[\begin{array}{ccc|c} . & . & . & 0 \\ . & 1 - A' & . & 0 \\ . & . & . & 0 \\ \hline 0 & 0 & 0 & 1 - \alpha \end{array} \right]$$

with $0 < \alpha < 1$ and a block-stochastic matrix A' not invertible.

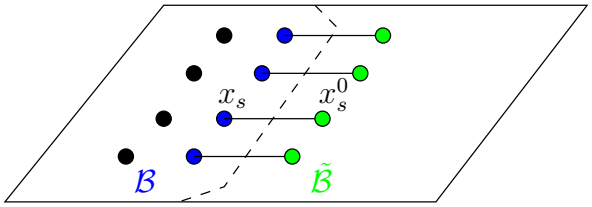


Figure 1: Von Neumann boundary conditions

3.2.2 Von Neumann boundary conditions

To each node $x_s \in \mathcal{B}$ we associate a new node x_s^0 connected to s only (see Fig. (1)). We note the related set of sites (states) as:

$$\tilde{\mathcal{B}} = \{x_s^0\}_{x_s \in \mathcal{B}}$$

Now define a new energy function

$$U(f) = \sum_{x_s \sim x_t} a_{s,t} \, (f(x_s) - f(x_t))^2 + \epsilon \sum_{s \in B} (f(x_s) - f(x_s^0))^2 \qquad (\epsilon > 0)$$

Minimizing wrt. x_s^0 yields:

$$f(x_s^0) = f(x_s) \ , \tag{13}$$

which in some sense is the discrete analogue of Von Neumann condition:

$$(\frac{\partial f}{\partial \vec{n}})_{x_s} = 0 \ .$$

We can take $0 < \epsilon \ll 1$ in order not to perturbate optimal value at site $x_s \in \mathcal{B}$:

$$f(x_s) = \frac{\sum_{x_t \sim x_s} a_{s,t} \, f(x_t) + \epsilon \, f(x_s^0)}{\sum_{x_t \sim x_s} a_{s,t} + \epsilon} \tag{14}$$

In this von Neumann case the matrix A is **positive stochastic** and **non-symmetric**:

$$\text{from (13) one has: } A_{x_s^0 x_s} = 1 \quad \text{, whereas } A_{x_s x_s^0} = \frac{\epsilon}{\sum_{x_t \sim x_s} a_{s,t} + \epsilon}$$

This corresponds to **reflection**. Since there is no need here to project on $\tilde{\mathcal{E}}$, we go back to the initial problem (7) with von Neumann boundary conditions using initial matrix A:

$$\boxed{(\mathrm{I} - A) \ f = \Phi = -\frac{\phi}{\gamma}} \tag{15}$$

In this case Dautray (1989) elegantly proves the following

Theorem (Dautray) 3 *if A is* **strictly positive and stochastic**, *then the series $\sum_{M\geq 0}(A^M)\ \Phi$ converges provided that $\Phi \in Im(I-A)$.*

Proof: from (5) the series $\sum_{M\geq 0}(P_0 A^M)(\Phi) - \mu(\Phi)$ converges, at geometrical speed, $\forall \Phi \in E$. Now it suffices that $\mu(\Phi) = 0$ to ensure the convergence of the series

$$\sum_{M\geq 0}(P_0 A^M)(\Phi) \quad \forall P_0$$

Lemma *previous condition: $\mu(\Phi) = 0$, is equivalent to $\Phi \in Im(I-A)$.*

a) indeed, if $\exists g \in E \ \ s.t. \ \ \Phi = g - Ag$, then $\mu(\Phi) = \mu(g) - (\mu A)(g) = 0$.

b) conversely, if $\mu(\Phi) = 0$ then, noting $g_n = \sum_{M=0}^{n} A^M \Phi$ we have that

$$P_0(g_n) = P_0(A^0\ \Phi + A\ g_{n-1}) \quad \forall P_0$$

and at the limit $P_0(g) = P_0(\Phi + Ag)$ *i.e.,* $P_0(\Phi) = P_0(g - Ag) \quad \forall P_0$.
In other words, if problem (15) admits a solution, then the series expansion: $\sum_{M\geq 0}(A^M)\ \Phi$ converges to its solution.
Easily generalized to A^r strictly positive for some $r > 1$:

Theorem 4 *Previous Theorem also holds if A is* **irreducible** i.e., *there exists a positive integer: $r > 1$ s.t. $A^r > 0$.*

Proof: indeed since the Markov chain with transition matrix A is **irreducible** and **positive recurrent** *i.e.,* the return time

$$T_s = \min\{\ n \geq 1\ s.t.\ X_n = x_s \mid X_0 = x_s\ \}$$

to any state x_s has **finite expectation**: indeed any state x_s can be reached again at every even step (nearest-neighbour interaction), then Wikipedia on the Web (2002); Fort *et al.* (2005)) this chain admits a unique invariant measure μ, which is also the invariant, **positive** measure μ_r of matrix $A^r > 0$ since:

$$\mu\ A = \mu \Rightarrow \mu\ A^r = \mu \quad \forall r \geq 1$$

Now from (6) with $M = nr + p,\ 0 \leq p < r$:

$$|P_0\ A^{nr+p}(\Phi) - \mu\ A^p(\Phi)| \leq c(A^r)^n\ c(A)^p\ ||\ P_0\ - \mu\ ||_V\ ||\ \Phi\ ||_\infty \qquad 0 \leq p < r$$

so that it suffices that $\mu(\Phi) = 0 = \ldots \mu\ A^p(\Phi)$ to ensure the "stepwise-geometrical" [3] convergence of the series with term $P_0\ A^M(\Phi)$.

[3] of course the convergence will be slower that previously, depending on r which may be large: of the order of the diameter of the lattice for the laplacian case.

3.3 interpretation in terms of Markov chains

We saw in previous sections that in many cases the matrix A is **stochastic** and hence the series expansion $\tilde{A}^M_{x\,y}$ can be interpreted as a transition probability for the Markov chain with transition matrix $\tilde{A}$. Anyway, since we need to consider absorbing states (see afterwards: Dirichlet conditions and attachment to data terms in image restoration), it could be dangerous to modify A (or $\tilde{A}$) for tailoring them to a given application.
Therefore the best way is given by Rubinstein (1981)[4] and Ycart (1997): it consists to design a specific Markov chain with initial probability P_0 adapted to our solution and whose transition matrix Q is as close as possible to $\tilde{A}$.
For instance consider the problem (8). The solution being f and given measure a μ on $\mathcal{E}$, we want to evaluate:

$$\mu(f) = \sum_{x_0 \in \mathcal{E}} \mu(x_0)\ f(x_0)$$

We consider for this purpose a Markov chain on

$$\mathcal{E} \cup \{\mathrm{a}\}$$

where a is a new, **absorbing** state.
This chain being specified by (P_0, Q), we consider the random variable

$$Z' = \frac{\mu(x_0)}{P_0(x_0)} \left(\prod_{X_{M+1} \neq a} \frac{\tilde{A}_{X_M\, X_{M+1}}}{Q_{X_M\, X_{M+1}}} \right) \frac{\psi(X_\tau)}{Q_{X_\tau\, a}} \tag{16}$$

It must be understood here that a is the first absorbing state encountered, and at step $\tau + 1$ *i.e.,*

$$X_{M+1} \neq X_{\tau+1} = a \ \ \forall M < \tau \ .$$

The theory states that whatever the Markov chain considered,

$$-\mu(f) = \ \mathbb{E}\,[\,Z'\,] \quad \textit{w.r.t.}\ \text{the chosen Markov chain}$$

(the sign arises from (12)). This expression can be approximated by the empiric average

$$-\mu(f) = \ \left(\sum_{i=1}^{N^s} Z' \right) /\ N^s$$

obtained over N^s simulations of this Markov chain. In practice one chooses

$$\begin{cases} P_0(x_0) & = \mu_{x_0} \\ Q_{x_s\, x_t} & = \tilde{A}_{x_s\, x_t} \quad x_s, x_t \in \tilde{\mathcal{E}} \\ Q_{y\, a} & = 1 \end{cases}$$

The only difference being that a is an absorbing state for the Markov chain (P_0, Q), *e.g.* linked to a Dirichlet boundary value or to a data node, so that:

$$Z' = \psi(X_\tau) \quad s.t. \ \ X_{\tau+1} = a$$

Thus the solution is estimated by the empiric average of values at nodes (states) connected to absorbing state a [5]

[4] this is among the first instances of sequential importance sampling: see *e.g.* Liu (2001) .

[5] it is fundamental to note the **backward** aspect of these equations: for instance y is treated as a sink here whereas it is obviously a "source". This relates to backward Kolmogorov-Chapman *versus*

3.4 summary

	Neumann	Dirichlet
$Q =$	$\begin{array}{c} \\ \tilde{\mathcal{E}} \\ \mathcal{B} \\ \tilde{\mathcal{B}} \end{array} \begin{array}{c} \begin{array}{cccc} \tilde{\mathcal{E}} & & \mathcal{B} & & \tilde{\mathcal{B}} \end{array} \\ \left[\begin{array}{cc\|cc\|c} . & . & . & . & 0 \\ . & . & . & . & 0 \\ \hline . & . & . & . & . \\ . & . & . & . & . \\ \hline 0 & 0 & 1 & . & 0 \end{array}\right] \end{array}$	$\begin{array}{c} \\ \tilde{\mathcal{E}} \\ \mathcal{B} \\ a \end{array} \begin{array}{c} \begin{array}{cccc} \tilde{\mathcal{E}} & & \mathcal{B} & & a \end{array} \\ \left[\begin{array}{cc\|cc\|c} . & . & . & . & 0 \\ . & . & . & . & 0 \\ \hline . & . & . & . & . \\ . & . & . & . & . \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$

Table 1: The specification of matrix Q according to boundary conditions

3.5 three ways to understand "time" in diffusion processes

This section is inspired from Dautray (1989). Let us consider a Markov chain with initial measure P_0 and stochastic matrix Q defined as in previous Section, according to the problem considered (Dirichlet or Neumann conditions - Poisson or Laplace). One has:

$$\begin{aligned} & \forall M \geq 0, \;\; P_M = P_0 \; Q^M \quad \textit{i.e.,} \quad \forall M \geq 0, \;\; P_{M+1} = P_M \; Q \\ \Leftrightarrow & \forall M \geq 0, \;\; P_{M+1} - P_M = P_M \; (Q - \mathrm{I}) \end{aligned} \tag{17}$$

Previous equation applies to the linear form $P_M \in E^*$ (the dual space of E).
We turn it into a scalar equation by applying it to any function $g \in E$:

$$\begin{aligned} P_{M+1}(g) - P_M(g) &= (P_M \; (Q - \mathrm{I}))\,(g) = P_M\,((Q - \mathrm{I}) \; g) \\ \mathbb{E}_{M+1}\,[\,g\,] - \mathbb{E}_M\,[\,g\,] &= \mathbb{E}_M\,[\,(Q - \mathrm{I}) \; g\,] \\ \Leftrightarrow \mathbb{E}\,[\,g(X_{M+1})\,] - \mathbb{E}\,[\,g(X_M)\,] &= \mathbb{E}\,[\,((Q - \mathrm{I}) \; g)\,(X_M)\,] \end{aligned} \tag{18}$$

Here three different possibilities occur:

1. assigning $g(x) = e_y(x) = \mathbb{1}_{x=y}$ and assuming Q symmetric (as for Dirichlet case):

$$\underbrace{\frac{P(X_{M+1} = y) - P(X_M = y)}{\dfrac{\partial P(X_t = y)}{\partial t}}}_{} = \underbrace{\mathbb{E}\,[\,((Q - \mathrm{I})\;)\,\mathbb{1}_{X_M = y}\,]}_{\frac{1}{\gamma}\;\Delta P(X_t = y)}$$

   This is the analogue of the heat equation for $P(X_t)$ where here time $t = M$ *i.e.,* is equal to the current Markov chain step. Of course, this applies (by identification) to all conditional probabilities knowing initial condition $X_0 = x_0$:

$$\frac{\partial P(X_t = y \mid X_0 = x_0)}{\partial t} = \frac{1}{\gamma} \; \Delta P(X_t = y \mid X_0 = x_0)$$

Fokker-Planck forward equations.

2. assigning g s.t. $(Q-\mathrm{I})\ g(x) = \dfrac{1}{\gamma}\ \mathbb{1}_{x=y}$: namely g is solution of the Poisson problem, and thus indeed the previous Green function $g(x) = \mathcal{G}(x,\ y) = G_{x\,y}$, one has:

$$\frac{\underbrace{\mathbb{E}\,[\,\mathcal{G}(X_{M+1},\ y)\,] - \mathbb{E}\,[\,\mathcal{G}(X_M,\ y)\,]}}{\dfrac{\partial \mathbb{E}\,[\,\mathcal{G}(X_t,\ y)\,]}{\partial t}} = \frac{1}{\gamma}\ \underbrace{P(X_M = y)}_{P(X_t = y)}$$

3. assigning $g = f$ s.t. $(Q - \mathrm{I})\ f = \dfrac{1}{\gamma}\ \psi$, namely $g = f$ is a solution of the Laplace problem, and we obtain similarly to previous case:

$$\frac{\underbrace{\mathbb{E}\,[\,f(X_{M+1})\,] - \mathbb{E}\,[\,f(X_M)\,]}}{\dfrac{\partial \mathbb{E}\,[\,f(X_t)\,]}{\partial t}} = \frac{1}{\gamma}\ \underbrace{\mathbb{E}\,[\,\psi(X_M = y)\,]}_{\mathbb{E}\,[\,\psi(X_t)\,]}$$

This applies by identification to every conditional probability knowing $X_0 = x_0$:

$$\frac{\partial \mathbb{E}\,[\,f(X_t) \mid X_0 = x_0\,]}{\partial t} = \frac{1}{\gamma}\ \mathbb{E}\,[\,\psi(X_t) \mid X_0 = x_0\,]$$

Back to discrete analysis, let us assume **finite** time τ to reach boundary $\mathcal{B}$ from x_0. Then one has $X_{M+1} = X_M\ (= a)$ as soon as $M > \tau$.
Summing down (!) the first τ steps in the Dirichlet case leads to:

$$\begin{aligned}
\mathbb{E}\,[\,f(X_\tau) \mid X_0 = x_0\,] - \mathbb{E}\,[\,f(X_{\tau-1}) \mid X_0 = x_0\,] &= \frac{1}{\gamma}\ \mathbb{E}\,[\,\phi(X_{\tau-1}) \mid X_0 = x_0\,] \\
\ldots \qquad - \qquad \ldots \qquad &= \frac{1}{\gamma}\ \mathbb{E}\,[\,\phi(X_M) \mid X_0 = x_0\,] \\
\mathbb{E}\,[\,f(X_1) \mid X_0 = x_0\,] - \mathbb{E}\,[\,f(X_0) \mid X_0 = x_0\,] &= \frac{1}{\gamma}\ \mathbb{E}\,[\,\phi(X_0) \mid X_0 = x_0\,]
\end{aligned}$$

$$\Rightarrow f(x_0) = -\ \frac{1}{\gamma}\ \left[\ \sum_{0 \le M < \tau} \mathbb{E}\,[\,\phi(X_M) \mid X_0 = x_0\,]\ \right] + \mathbb{E}\,[\,b(X_\tau)_{X_\tau \in \mathcal{B}} \mid X_0 = x_0\,]$$

This in full accordance with (3) and (16) for $x_0 \notin \mathcal{B}$: :

$$\begin{aligned}
f(x_0) = \tilde{f}(x_0) &= -\ \frac{1}{\gamma}\ \sum_{0 \le M \le \tau} (Q^M\ \psi)(x_0) \\
&= -\ \frac{1}{\gamma}\ \left[\ \sum_{0 \le M < \tau} \mathbb{E}\,[\,\phi(X_M) \mid X_0 = x_0\,]\ \right] +\ \mathbb{E}\,[\,B(X_\tau) \mid X_0 = x_0\,]
\end{aligned}$$

Thus in practice, simulate a series of N_{x_0} uniform random walks starting from x_0 and terminating when reaching domain boundary Dautray (1989):

$$f(x_0) \approx \frac{1}{N_{x_0}}\ \sum_{n=1}^{N_{x_0}}\ \left(\ -\frac{1}{\gamma}\ \left[\ \sum_{M < \tau^{(n)}} \phi(X_M^{(n)})\ \right] +\ B(X_{\tau^{(n)}})_{X_{\tau^{(n)}} \in \mathcal{B}}\ \right)$$

The Poisson problem is treated in next Section.

4 Application to path routing in *ad hoc* networks

We show in this Section that previous results are related to modeling the number of paths arriving to a given point in a regular *ad hoc* network. During the course of this work we became aware of the excellent work of Mabrouki *et al.* (2008b,a); Mabrouki (2008). Interesting developments on *ad hoc* networks are also found in Hsin and Liu (2008); Beraldi (2009). An application on Internet page finding is found in Ollivier and Senellart (2007).

The case of Poisson problem

a) First we want to "simulate" $G_{x_0\,y}$ given by previous results.
This corresponds to assign $\mu(x) = \mathbb{1}_{x=x_0}$, and we take thus:

- $P_0(x) = \mu(x) = \mathbb{1}_{x=x_0}$.
- the absorbing state a is related to "sink" node y .
- boundary conditions are accounted for in the following way:

- Neumman conditions:
  since the optimal solution should verify: $f(x_s^0) = f(x_s)$, we assign:

$$Q_{x_s^0, x_s} = 1 \quad (\textbf{reflection})$$

- Dirichlet condition (null/ non null):
  we assign the same absorbing state a to each $x_s \in \mathcal{B}$.

From (16) we have simply that:

$$Z' = \mathbb{1}_{X_M=y \;|\; X_0=x_0}$$

so that the solution (9) for null Dirichlet boundary conditions writes as:

$$\begin{aligned} -4\; G_{x_0\,y} &= \sum_{M\geq 0} \mathbb{E}\left[\, \mathbb{1}_{X_M=y \;|\; X_0=x_0} \,\right] \\ &= \mathbb{E}\left[\, \mathbb{1}_{exists\ M\geq 0\ s.t.\ X_M=y \;|\; X_0=x_0} \,\right] \\ &= Pr\ (\exists \text{ path: } x_0 \rightsquigarrow y) \quad !! \end{aligned}$$

The last expression can be approximated by its empiric value

$$Pr\ (\exists \text{ path: } x_0 \rightsquigarrow y) \approx \frac{N_{x_0,y}}{N_{x_0}}$$

where $N_{x_0,y}$ counts the number of random paths with origin x_0 arriving to y among the N_{x_0} simulated random paths with origin x_0.
Hence the discrete approximation of the Green function of the Poisson problem

$$\Delta_x G(x,y) = \delta(x,y)$$

with null Dirichlet absorption conditions at boundary counts, up to sign and constant, the number of paths [6] going from x to y .

b) Another application is the following: consider choosing

$$P_0 = \mu_0$$

(*i.e.,* . the starting point of the Markov chain is chosed at random uniformly in $\mathcal{E}$). Since the solution of the Poisson problem is $f(x_0) = G_{x_0\,y} \quad \forall x_0 \in \mathcal{E}$, this yields:

$$\mu_0(f) = -\frac{\sum_{x_0 \in \mathcal{E}} G_{x_0\,y}}{(\ N = |\mathcal{E}|\ )} = \frac{1}{4}\ Pr\ (\exists \text{ path} \quad \rightsquigarrow\ y) \approx \frac{1}{4}\ \frac{N^y}{N^s}$$

where N^y counts the number of paths arriving to the (absorbing) state y among the N^s simulated paths starting at random (uniform) in $\mathcal{E}$.
A result for the continuous case (see Appendix A) shows that for a circular zone of radius R and with absorbing conditions at the boundary, $\int G(x,y)\ d^2x = \frac{r^2 - R^2}{4}$ where point y is at distance r from center O . This has been used to prove Perreau's and Pham formula Pham and Perreau (2002, 2004) for the average traffic passing through a given node in a circular zone with uniform density of mobiles [7].
Thus for fixed radius R and at the "continuous limit" *i.e.,* $N \to +\infty \Leftrightarrow h \to 0$:

$$-\frac{\sum_{x_0 \in \mathcal{E}} G_{x_0\,y}}{N} = \frac{1}{4}\ P(\exists \text{ path:} \quad \rightsquigarrow\ y)$$

$$\underset{N \to +\infty}{\longrightarrow} -\ \frac{\int G(x,y)\ d^2x}{\pi\ R^2} = \frac{1}{4\ \pi}\ (1 - \frac{r^2}{R^2})$$

and thus

$$P(\exists \text{ path:} \quad \rightsquigarrow\ y) \underset{N \to +\infty}{\longrightarrow} \frac{1}{\pi}\ (1 - \frac{r^2}{R^2}) \tag{19}$$

A curious consequence : the probability that any path (with absorbing conditions at the boundary) arrives to the center of the circle is

$$\frac{1}{\pi}\ .$$

[6] Notice that the first terms $M < ||x_0 - y||_1$ (the L^1 norm) are null since no transitions occurs from x_0 to y in less than $||x_0 - y||_1$ steps.

[7] at least its analytical form as a function of the distance of the node to the center.

5 Application to image restoration

Notice first that similar algorithms for image segmentation have been investigated in Grady (2008); Grady and Sinop (2008); Sinop and Grady (2007); Grady (2006).
Let as before Φ denote the observed image data. The lattice step h is taken equal to 1. The posterior restoration energy is then

$$U(f) = \sum_{x_s \sim x_t} a_{s,t} \, (f(x_s) - f(x_t))^2 + \beta \, \sum_{s \in B} (f(x_s) - \phi(x_s))^2 \tag{20}$$

The connectivity is noted γ . Assume also free (or Von Neumann) boundary conditions.

5.1 simplest case : no boundary process

This corresponds to assign

$$a_{s,t} = 1 \quad \forall x_s \sim x_t$$

The optimal solution verifies then:

$$f(x_s) = \frac{\sum_{t \sim s} f(x_t) + \beta \, \phi(x_s)}{\gamma + \beta} \tag{21}$$

$$\Rightarrow f = A \, f + \frac{\beta}{\gamma + \beta} \, \phi \tag{22}$$

$$\text{with } A_{x_s \, x_t} = \frac{1}{\gamma + \beta} \quad \forall s, t \in \mathcal{E} \tag{23}$$

Assuming $\beta > 0$ safely implies the following series expansion:

$$f = \sum_{M \geq 0} A^M \, \frac{\beta}{\gamma + \beta} \, \phi \tag{24}$$

$$f(x_s) = \sum_{M \geq 0} \sum_{x_t \in \mathcal{E}} (A^M)_{x_s \, x_t} \, \frac{\beta}{\gamma + \beta} \, \phi(x_t) \quad \forall x_s \in \mathcal{E} \tag{25}$$

It is useful for the simulated interpretation to write

$$\phi(x_s) = \phi(y_s) \quad \forall x_s \in \mathcal{E}$$

where y_s is the supplementary (data) node associated to x_s (see Fig(2))

$$\text{and to assign: } A_{x_s \, y_s} = \frac{\beta}{\gamma + \beta}$$

Previous equations write then

$$f(x_s) = \sum_{M \geq 0} \sum_{x_t \in \mathcal{E}} (A^M)_{x_s \, x_t} \, A_{x_t \, y_t} \, \phi(y_t) \quad \forall x_s \in \mathcal{E} \tag{26}$$

Thus considering all observed nodes y_s as **absorbing states** we obtain the algorithm described in Table 1.

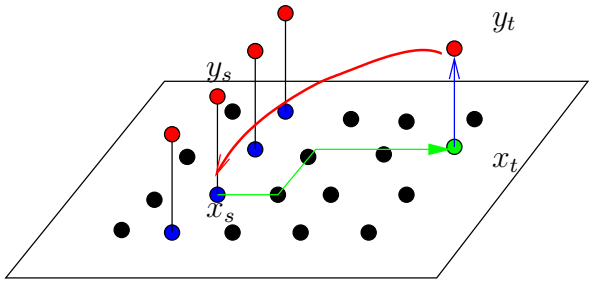


Figure 2: Image restoration

Algorithm 1 Algorithm for image restoration by stochastic diffusion (without boundaries)

for each site s **do**
  for $k = 0$ to N^s times **do**
    while (random uniform $\rho \in [0,1]) < \dfrac{\gamma}{\gamma+\beta}$ **do**
      random 2D walk: $x_s \rightarrow x_i \rightarrow x_{i+1} \rightarrow \ldots \rightarrow x_t$
        (using transition probability $A_{x_i\, x_{i+1}}$)
    end while
    as soon as $\rho \geq \dfrac{\gamma}{\gamma+\beta}$ (transition $x_t \rightarrow y_t$):
    $\leftarrow$ pick up observed value $\phi(y_t)$
  end for
  $f(x_s) \leftarrow$ empiric average of these N^s picked up values.
end for

Let us also show that the overall average length (*i.e.,* over the number of steps) of simulated random walks is:

$$< L >= \mathbb{E}\,[\,M\,] = \frac{\gamma}{\beta}$$

A simple way to prove this is to consider the following auxiliary Markov chain of binary "lifetime" variables specified by

$$\begin{cases} P(\xi_{i+1}=1 \mid \xi_i = 1) & = \alpha = \dfrac{\gamma}{\gamma+\beta} \quad \text{"lifetime prolongation"} \\ P(\xi_{i+1}=0 \mid \xi_i = 0) & = 1 \qquad\qquad\quad \text{"irreversible decay"} \end{cases}$$

Since $A_{x_t\ y_t} = 1-\alpha = \dfrac{\beta}{\gamma+\beta}$, one can write:

$$f(x_s) = \sum_{x_t \in \mathcal{E}} \sum_{M \geq 0} \underbrace{\left(\frac{\gamma}{\gamma+\beta}\right)^M}_{\alpha^M} \left(\frac{1}{\gamma}\right)^M (1-\alpha)\ \phi(x_t)$$

$$= \sum_{x_t \in \mathcal{E}} \sum_{M \geq 0} \underbrace{P(\xi_M = 1,\ \xi_{M+1}=0 \mid \xi_0 = 1)}_{\text{decay at time } M} \underbrace{\left(\frac{1}{\gamma}\right)^M}_{\text{"free random walk" of } M \text{ steps}} \phi(x_t)$$

$$= \sum_{x_t \in \mathcal{E}} \sum_{M \geq 0} \mathbb{E}\,[\,\mathbb{1}_{\xi_M=1,\ \xi_{M+1}=0} \times \phi(X_M = x_t)\,]$$

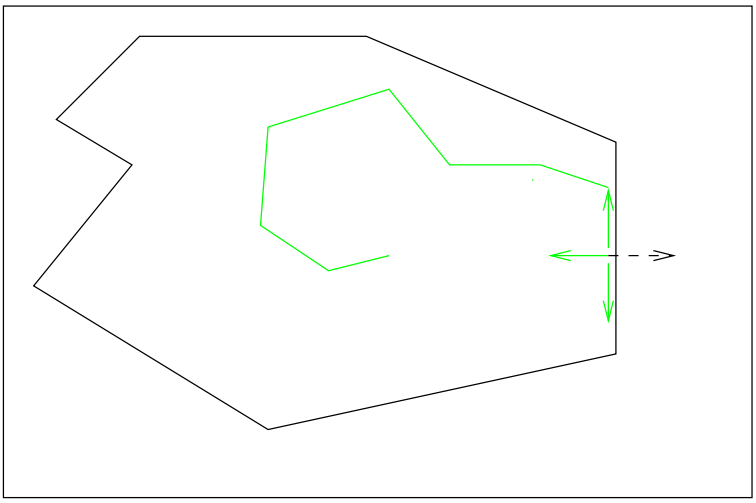


Figure 3: Constrained diffusion with boundary process

The expected lifetime of the "particle" ξ is classically:

$$\frac{\alpha}{1-\alpha} = \frac{\gamma}{\beta} \qquad \text{QED} \ .$$

5.2 adding a boundary process

Assume that image boundary localization is known. The $a_{s,t}$ are then specified as:

$$a_{s,t} = \begin{cases} 1 & \text{if no boundary} \quad \text{between } x_s \text{ and } x_t \\ 0 & \text{if boundary} \quad\ \ \text{between } x_s \text{ and } x_t \end{cases}$$

We obtain similar equations to (14) with ϵ replaced by β and x_s^0 by y_s :

$$f(x_s) = \frac{\sum\limits_{x_t \sim x_s} a_{s,t}\ f(x_t) + \beta\ f(y_s)}{\sum\limits_{x_t \sim x_s} a_{s,t} + \beta} \tag{27}$$

These equations can be interpreted by defining the *effective* connectivity at site x_s:

$$\Gamma_s = \sum_{x_t \sim x_s} a_{s,t} \quad \text{so that:}$$

$$\begin{cases} A_{x_s\, x_t} &= \dfrac{a_{s,t}}{\Gamma_s + \beta} \\ A_{x_s\, y_s} &= \dfrac{\beta}{\Gamma_s + \beta} \end{cases}$$

This corresponds to constrained diffusion in the image (see Fig. 3): pick up only contributions inside the region to which the investigated pixel x_s belongs.
The corresponding algorithm is described in Table 2.

Algorithm 2 Algorithm for image restoration by stochastic diffusion (with boundaries)

for each site s **do**
  for $k = 0$ to N^s times **do**
    while (random uniform $\rho \in [0,1]) < \dfrac{\Gamma_i}{\Gamma_i + \beta}$ **do**
      random 2D walk: $x_s \rightarrow x_i \rightarrow x_{i+1} \rightarrow \ldots \rightarrow x_t$
        (using transition probability $A_{x_i\, x_{i+1}}$)
    end while
    as soon as $\rho \geq \dfrac{\Gamma_t}{\Gamma_t + \beta}$ (transition $x_t \rightarrow y_t$):
    $\leftarrow$ pick up observed value $\phi(y_t)$
  end for
  $f(x_s) \leftarrow$ empiric average of these N^s picked up values.
end for

5.3 experimental results

We created a synthetic mire image of size 160×160 composed of four piecewise-constant regions with respective grey levels 50, 100, 150 and 200. Then we superimposed additive gaussian noise with STD $\sigma = 30$ and $\sigma = 50$. Restoration results are shown in figures 4 and 5 for number of simulated random walks $N^s = 256$.
Fig. 6 shows instances of walks starting from a given pixel.

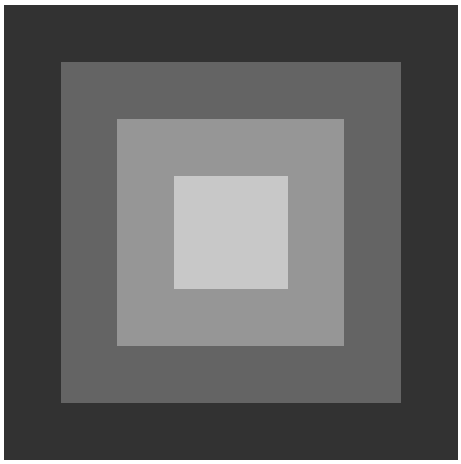
6 Conclusion

This document has been intended to present in a completely discrete framework and in a simple way if possible the complex theory of diffusion processes, which is itself linked to stochastic differential equations (Ito calculus, Brownian processes) and its intrinsic relationship to partial differential equations Dautray (1989); Ycart (1997).
We have tried to show two applications, namely in image restoration and in traffic path routing for *ad hoc* networks.
For further in-depth investigation it is now needed:

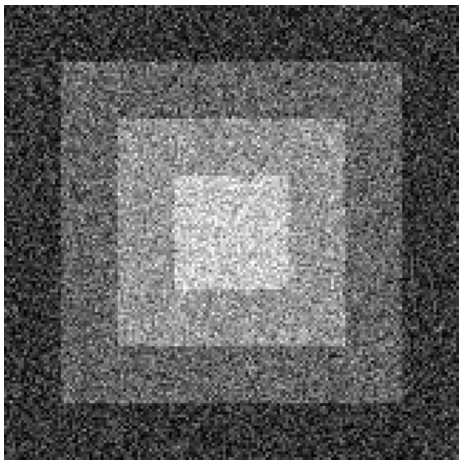
- to establish a "flow equation" more adapted to realistic traffic routing mechanism between two points in a given network and derive the "ensemble" equations associated to all couples of such nodes.
- to re-formulate the well-known image restoration by anisotropic diffusion framework Perona and Malik (1990); Catte *et al.* (1992); Wei (1999) and even also non-local image restoration A. Buades (2005, 2006) using this approach.
- to extend the adaptive (and iterative) algorithms enabling both pixel and boundary processes estimation in image restoration such as the LEGENDRE and ARTUR algorithms Charbonnier *et al.* (1993); Charbonnier (1994).

To conclude, this is not very difficult; one just needs to know: ‘‘`de tout un peu`’’ [8] !

[8] "a little bit of everything" .

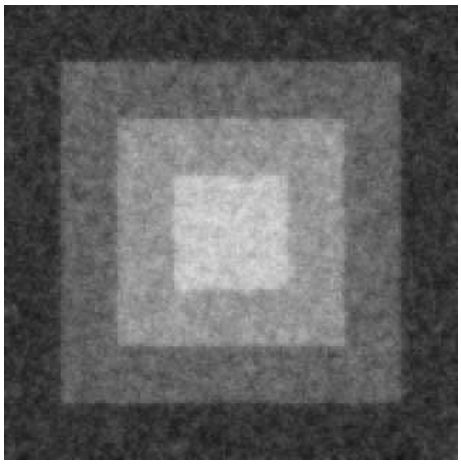


original image

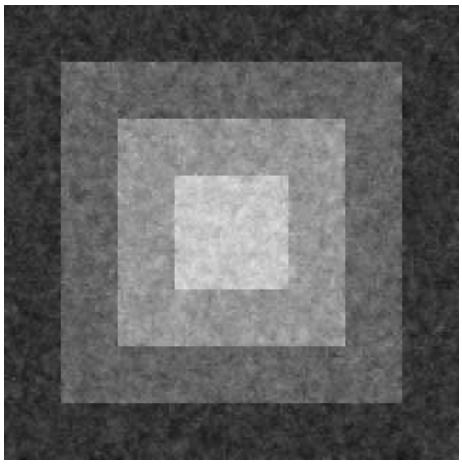


noisy image

Datas

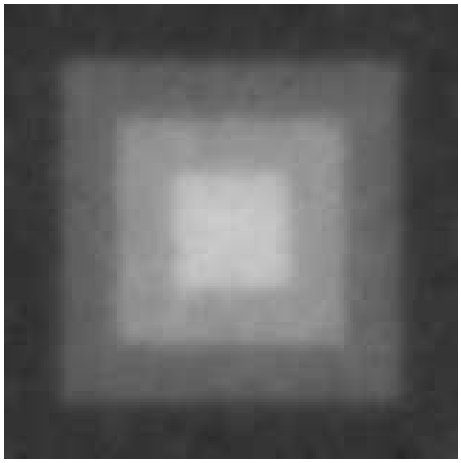


no boundaries

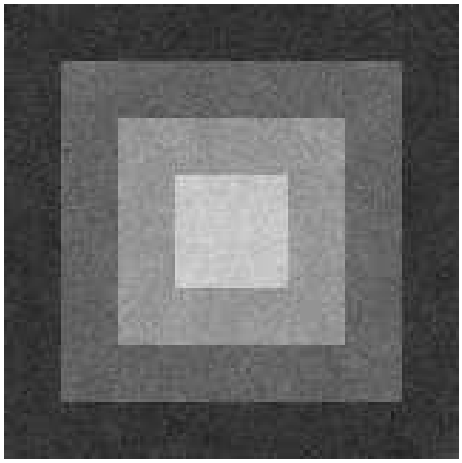


with boundaries

$\beta = 1$ (average free diffusion length = 4)



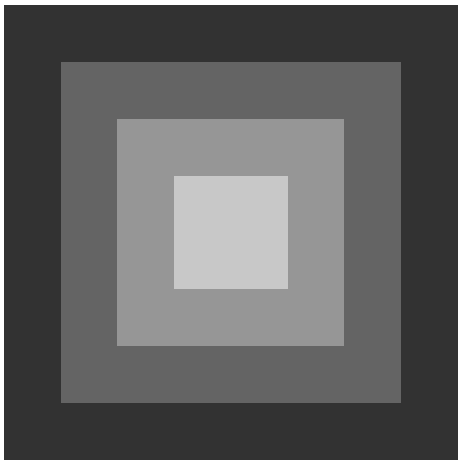
no boundaries



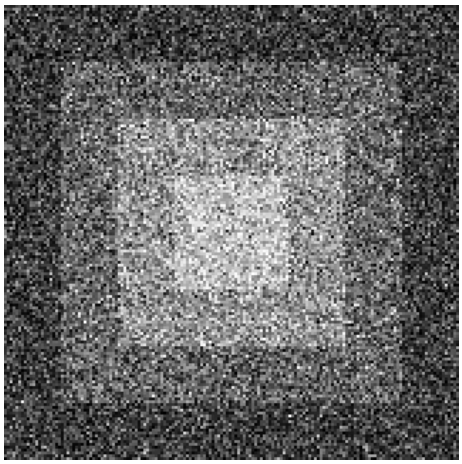
with boundaries

$\beta = 0.1$ (average free diffusion length = 40)

Figure 4: Restoration results for noisy image with $\sigma = 30$

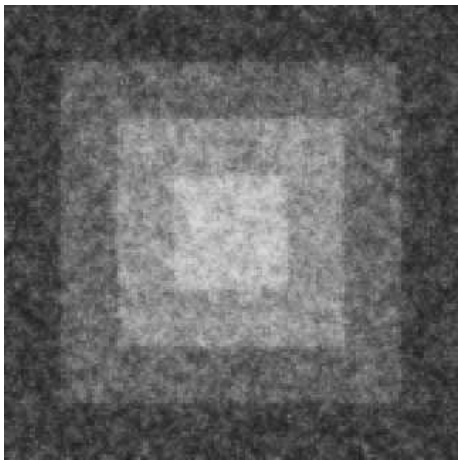


original image

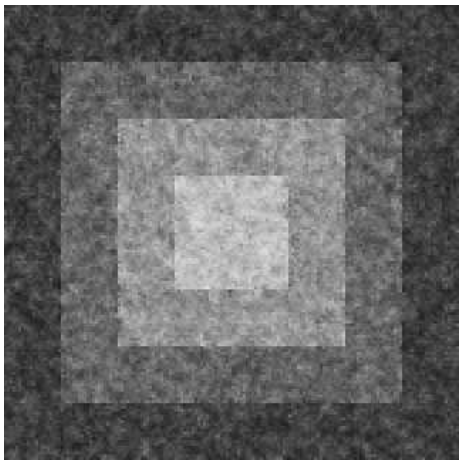


noisy image

Datas

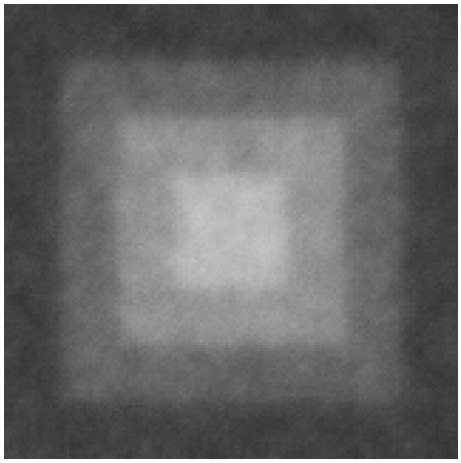


no boundaries

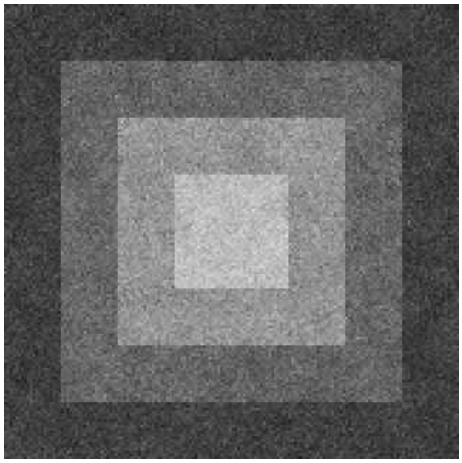


with boundaries

$\beta = 1$ (average free diffusion length = 4)



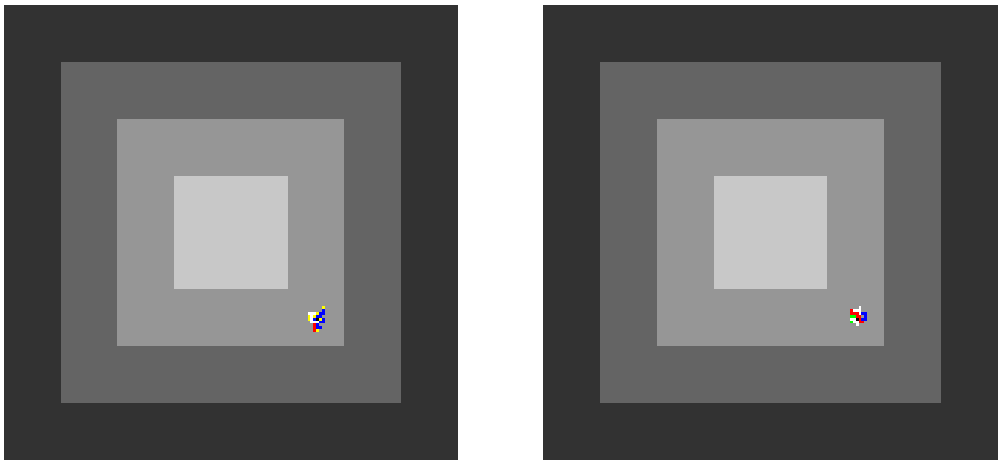
no boundaries



with boundaries

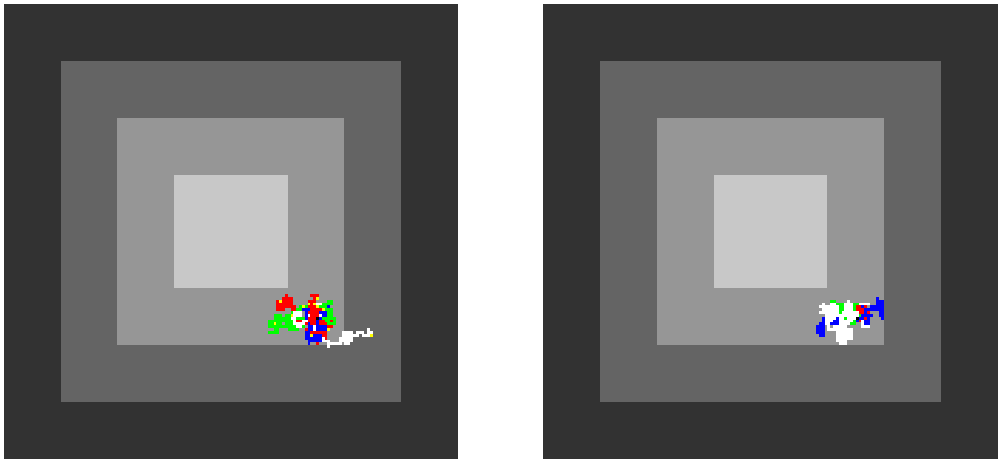
$\beta = 0.1$ (average free diffusion length = 40)

Figure 5: Restoration results for noisy image with $\sigma = 50$



no boundaries with boundaries

$\beta = 1$ (average free diffusion length = 4)



no boundaries with boundaries

$\beta = 0.1$ (average free diffusion length = 40)

Figure 6: Various random walks starting from $O = (110, 110)$

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A A note on Green functions on the circle

A.1 Introduction and notations

In this appendix we try to explicite and summarize well-known properties of Green functions in 2-dimensional domains. This gives place to a first formulation of the number of paths going through a given point in a circular zone for traffic estimation purposes in *ad hoc* networks Pham and Perreau (2002).
Our notations are the following: we shall denote by $\mathbf{x}, \mathbf{y}, \mathbf{z}$ some vectors (or points) of $\mathbb{R}^2$ and by $|\mathbf{x}|$ their euclidian norms.

A.2 The third Green formula

Can be seen as a "vectorial wronskian" [9] formulation:

$$\mathrm{div}\left(F(\mathbf{x})\ \vec{\nabla}G(\mathbf{x}) - G(\mathbf{x})\ \vec{\nabla}F(\mathbf{x})\right) = F(\mathbf{x})\ \Delta G(\mathbf{x}) - G(\mathbf{x})\ \Delta F(\mathbf{x}) \tag{28}$$

so that by the Green-Ostrogradsky formula one obtains:

$$\begin{aligned}\int\!\!\int_D &\left(F(\mathbf{x})\ \Delta G(\mathbf{x}) - G(\mathbf{x})\ \Delta F(\mathbf{x})\right)\ \mathrm{d}\mathbf{x} \\ &= \int_{\partial D}\left(F(\mathbf{x})\ \vec{\nabla}G(\mathbf{x}) - G(\mathbf{x})\ \vec{\nabla}F(\mathbf{x})\right)\ .\ \vec{n}\ \mathrm{d}s(\mathbf{x})\end{aligned} \tag{29}$$

where the right part member of the equation means the *flux* of the associated vector field on the boundary of domain D.

A.3 Green functions - Laplace and Poisson cases

It is easy to check that in 2D one has:

$$\Delta_{\mathbf{x}} \log|\mathbf{x}-\mathbf{y}| = 2\pi\ \delta(\mathbf{x},\ \mathbf{y})\ \ \forall \mathbf{x}, \mathbf{y} \in D$$

where $\Delta_{\mathbf{x}}$ means that the Laplacian is taken with respect to the variable $\mathbf{x}$.
As a matter of fact let us consider a small disk D_ρ of radius ρ centered on $\mathbf{y}$. First Green

[9] indeed one can consider that

$$\mathrm{div}\begin{vmatrix} F(\mathbf{x}) & G\mathbf{x}) \\ \vec{\nabla}F(\mathbf{x}) & \vec{\nabla}G(\mathbf{x}) \end{vmatrix} = \begin{vmatrix} F(\mathbf{x}) & G(\mathbf{x}) \\ \Delta F(\mathbf{x}) & \Delta G(\mathbf{x}) \end{vmatrix}$$

formula

$$\int\int_{D_\rho} \Delta_{\mathbf{x}} \log|\mathbf{x}-\mathbf{y}| \ \mathrm{d}\mathbf{x} = \int_{\partial D_\rho} \vec{\nabla} \log|\mathbf{x}-\mathbf{y}| \ . \ \vec{n} \ \mathrm{d}s(\mathbf{x}) = \int_{\theta=0}^{2\pi} \frac{1}{\rho} \ \rho \ \mathrm{d}\theta = 2\pi$$

Any function $\mathcal{G}(\mathbf{x},\ \mathbf{y})$ of *two* points on D which satisfies the **Poisson** equation:

$$\Delta_{\mathbf{x}} \mathcal{G}(\mathbf{x},\ \mathbf{y}) = \delta(\mathbf{x},\ \mathbf{y}) \quad \forall \mathbf{x},\ \mathbf{y} \in D \tag{30}$$

is called a Green function for the Laplacian operator on domain D.
In all this approach we have to be cautious when defining the distribution $\delta(.,.)$. Previous (Poisson equation) means indeed that

$$\int\int_D F(\mathbf{x}) \ \Delta_{\mathbf{x}} \mathcal{G}(\mathbf{x},\ \mathbf{y}) \ \ \mathrm{d}\mathbf{x} = F(\mathbf{y}) \ \ \forall \mathbf{y} \in D \ \ \forall F \in \mathcal{F}$$

where $\mathcal{F}$ is some functional space to define
(also I do not know what occurs for $\mathbf{y} \in \partial D$ in previous definition)
This framework is especially of use for solving Laplace-like equations

$$\Delta F = u \quad \text{with} \quad \left\{ \begin{array}{ll} F(\mathbf{x}) & = f(\mathbf{x}) \quad \forall \mathbf{x} \in \partial D \ (Dirichlet) \\ \vec{\nabla}_{\mathbf{F}}(\mathbf{x}) & = \vec{g}(\mathbf{x}) \quad \forall \mathbf{x} \in \partial D \ (von\ Neumann) \end{array} \right. \tag{31}$$

As a matter of fact, for the the simple case of Dirichlet boundary conditions
$F(\mathbf{x}) = 0 \ \ \forall \mathbf{x} \in \partial D$ one obtains from (29):

$$F(\mathbf{y}) \ = \int\int_D \Delta F(\mathbf{x}) \ \mathcal{G}(\mathbf{x},\ \mathbf{y}) \ \mathrm{d}\mathbf{x} = \int\int_D u(\mathbf{x}) \ \mathcal{G}(\mathbf{x},\ \mathbf{y}) \ \mathrm{d}\mathbf{x} \tag{32}$$

Here we see the "propagator" aspect of Green functions in the sense that they convey "information" from point $\mathbf{x}$ to point $\mathbf{y}$ via the factor $\mathcal{G}(\mathbf{x},\ \mathbf{y})$.

A.4 Green functions for the 2D circle

Our goal is to find a Green function $\mathcal{G}(\mathbf{x},\ \mathbf{y})$ on the disk domain C with radius R and with Dirichlet condition say, boundary value 0 on the circle $\Gamma = \partial$C.
Thus exactly as for Electrostatic Images we take the *harmonic conjugate* $\mathbf{y}'$ of $\mathbf{y}$, *i.e.*, its inverse for the inversion $\mathcal{I}$ of center $\mathbf{O}$ and power R^2, see Fig. 7). We check easily that since $\mathbf{y}'$ lies outside C:

$$\begin{aligned} \Delta_{\mathbf{x}}(\log|\mathbf{x}-\mathbf{y}| - \log|\mathbf{x}-\mathbf{y}'|) &= 2\pi \ (\ \delta(\mathbf{x},\ \mathbf{y}) - \delta(\mathbf{x},\ \mathbf{y}')\ ) \\ &= 2\pi \ \delta(\mathbf{x},\ \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathrm{C} \end{aligned} \tag{33}$$

Moreover, from the inversion $\mathcal{I}$ the triangles $\mathbf{Oxy}'$ and $\mathbf{Oyx}'$ are anti-similar, so that a forthcoming useful relationship writes:

$$\frac{|\mathbf{y}|}{|\mathbf{x}|} = \frac{|\mathbf{x}'-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}'|} \tag{34}$$

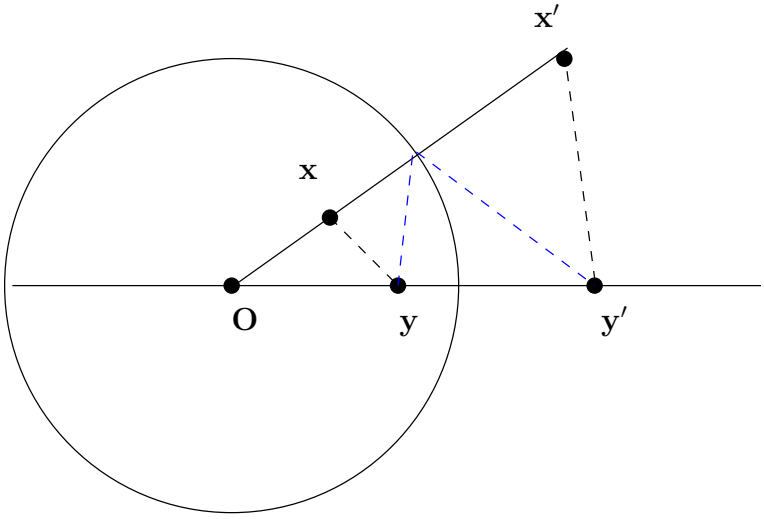

Figure 7: The Green function with null Dirichlet boundary conditions

In particular for $\mathbf{x} = \mathbf{x}' \in \mathrm{C}$ one has

$$\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}'|} = \frac{|\mathbf{y}|}{R} \quad \forall \mathbf{x} \in \mathrm{C} \tag{35}$$

so that

$$\mathcal{G}(\mathbf{x},\ \mathbf{y}) = \frac{1}{2\pi}\left(\log\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}'|} - \log\frac{|\mathbf{y}|}{R}\right) \tag{36}$$

is the announced Green function.
Also from (34) one sees that

$$\begin{aligned}\log\frac{|\mathbf{y}-\mathbf{x}|}{|\mathbf{y}-\mathbf{x}'|} - \log\frac{|\mathbf{x}|}{R} &= \log\frac{|\mathbf{y}-\mathbf{x}|}{|\mathbf{x}-\mathbf{y}'|}\frac{|\mathbf{x}|}{|\mathbf{y}|} - \log\frac{|\mathbf{x}|}{R} \\ &= \log\frac{|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}-\mathbf{y}'|} - \log\frac{|\mathbf{y}|}{R}\end{aligned} \tag{37}$$

Thus the Green function is **symmetric** although not it does not depend on $|\mathbf{x}-\mathbf{y}|$.

A.5 The Dirichlet problem on the circle

This classical problem consists in retrieving an harmonic real function in C from its boundary values on ∂C. From the third Green equation (29) one has

$$F(\mathbf{y}) = \int_{\partial D} F(\mathbf{x})\ \vec{\nabla}_{\mathbf{x}}\mathcal{G}(\mathbf{x},\ \mathbf{y})\ .\ \vec{n}\ \mathrm{d}s(\mathbf{x}) \tag{38}$$

From the formula (37) one has

$$F(\mathbf{y}) = \frac{1}{2\pi}\int_{\partial D} F(\mathbf{x})\ \left(\frac{\cos\phi}{|\mathbf{x}-\mathbf{y}|} - \frac{\cos\phi'}{|\mathbf{x}-\mathbf{y}'|}\right)\mathrm{d}s(\mathbf{x}) \tag{39}$$

Then from triangles relations we have:

$$|\mathbf{y}|^2 = R^2 + |\mathbf{x}-\mathbf{y}|^2 - 2\ R\ |\mathbf{x}-\mathbf{y}|\ \cos\phi$$
$$\Rightarrow \frac{\cos\phi}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{2\ R} + \underbrace{\frac{R^2-|\mathbf{y}|^2}{2\ R\ |\mathbf{x}-\mathbf{y}|^2}}_{A(\mathbf{x})}$$
$$|\mathbf{y}'|^2 = R^2 + |\mathbf{x}-\mathbf{y}'|^2 - 2\ R\ |\mathbf{x}-\mathbf{y}'|\ \cos\phi'$$
$$\Rightarrow \frac{\cos\phi'}{|\mathbf{x}-\mathbf{y}'|} = \frac{1}{2\ R} + \underbrace{\frac{R^2-|\mathbf{y}'|^2}{2\ R\ |\mathbf{x}-\mathbf{y}'|^2}}_{A'(\mathbf{x})}$$

Now it is very easy to check that $A'(\mathbf{x}) = -A(\mathbf{x})$ from the inversion property: $|\mathbf{y}'| = \dfrac{R^2}{|\mathbf{y}|}$

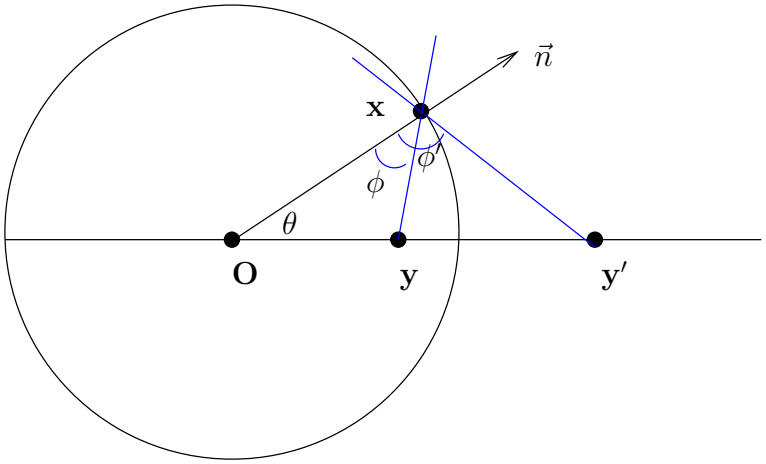

Figure 8: The Green function with non-null Dirichlet boundary conditions

and from relation (35), so that:

$$F(\mathbf{y}) = \frac{1}{2\pi}\int_{\partial D} 2A(\mathbf{x})\ \ \mathrm{d}\mathbf{x} = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} F(\mathbf{x})\ \frac{R^2-|\mathbf{y}|^2}{|\mathbf{x}-\mathbf{y}|^2}\ \mathrm{d}\theta \tag{40}$$

Of course this can be proved from Cauchy relation for analytic functions, whose real and imaginary components are harmonic functions.

A.6 Application to the number of paths going through some point $\mathbf{y} \in \mathbf{C}$

Suppose that we demonstrate that this number is

$$\mathcal{N}_{\mathbf{y}} = \int\!\!\int_D \rho\ \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \rho\ \mathcal{G}(\mathbf{y},\ \mathbf{z})\ \mathrm{d}\mathbf{x}\ \mathrm{d}\mathbf{z} \tag{41}$$

where ρ is the density of stations in our area. Then

$$\mathcal{N}_{\mathbf{y}} = \int\int_D \rho\, \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \mathrm{d}\mathbf{x} \times \int\int_D \rho\, \mathcal{G}(\mathbf{y},\ \mathbf{z})\ \mathrm{d}\mathbf{z} \tag{42}$$

$$= \rho^2 \left(\int\int_D \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \mathrm{d}\mathbf{x} \right)^2 \tag{43}$$

since the Green function is *symmetric*. Then consider the quantity

$$\gamma(\mathbf{y}) = \int\int_D \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \mathrm{d}\mathbf{x} \tag{44}$$

From (32) it is the solution of the Poisson equation

$$\Delta\gamma(\mathbf{y}) = 1 \quad \text{with} \quad \gamma(\mathbf{y}) = 0 \quad \forall \mathbf{y} \in \partial D$$

It is fairly simple to exhibit this solution:

$$\gamma(\mathbf{y}) = \frac{1}{4} \left(|\mathbf{y}|^2 - R^2 \right) \tag{45}$$

(since $\Delta(x^2 + y^2) = 4$). Thus

$$\mathcal{N}_{\mathbf{y}} = (\rho\, \gamma(\mathbf{y}))^2 = \frac{1}{16}\, \rho^2\, \left(|\mathbf{y}|^2 - R^2 \right)^2 \tag{46}$$

Quite similar to the result of Pham and Perreau (2002).

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