Theoretical Sound Field Analysis

Analyse de champs sonores théorique

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Résumé

L'objectif de l'analyse de champs sonores est de fournir une description des événements sonores non seulement dans le domaine temporel, mais aussi dans le domaine spatial. Cet article s'intéresse surtout à l'analyse de champs sonores en vue de son utilisation dans des systèmes de reproduction sonore telles que les techniques ambisonics ou de la Wave Field Synthesis. Les modèles de champs sonores utilisés dans ces deux approches appartiennent à deux catégories plus générales : les décompositions harmoniques d'une part, telles que les décompositions en ondes planes, harmoniques cylindriques et harmoniques sphériques, et les représentations basées sur l'équation intégrale de Kirchhoff d'autre part. Cet article présente une synthèse détaillée de ces différents modèles. Finalement, plusieurs descriptions du champ sonore sont utilisées pour modéliser le même événement sonore : celui d'ondes planes tronquées à l'intérieur d'une sphère d'un rayon donné. Les avantages et inconvénients de chacune de ces représentations sont alors discutés.

Theoretical sound field analysis

Mathieu Guillaume, Yves Grenier

May 2, 2006

Abstract

The analysis of sound fields aims at giving a description of sound events, not only in the time domain, but also in the space domain. This article is mainly focused on sound field analysis intended for sound reproduction systems such as ambisonics and Wave Field Synthesis. The sound field models used in these two approaches belong to two more general categories: harmonic decompositions, such as plane waves, cylindrical harmonics and spherical harmonics, and representations based on Kirchhoff's integral equation. This article presents a unified view of these models. Finally, several representations are compared to model the same sound field event, that is truncated plane waves inside a sphere of a given radius, and advantages and drawbacks of each representation are described.

1 Introduction

Sound field analysis is used in a variety of domains such as the study of vibrating stuctures [1], the accurate reproduction of a sound field over an extended area [2][3] and active noise control which is just a particular case of the latter, with a null reference field [4]. This paper is mainly concerned by sound field analysis applied to the reproduction of a sound field over an extended area. Under practical conditions, the sound field is sampled by a microphone array, and we can classify the sound field reproduction strategies according to whether these initial measures are directly processed, like in least-squares methods [5], or whether an intermediary spatial analysis step is carried out, such as in Ambisonics [3] or Wave Field Analysis/Synthesis [6][7] methods. In the first strategy, few works are concerned by the behavior of the sound field in the neighborhood of the sampling/control points [8]. In the second strategy, sound field analysis aims at giving an accurate representation of the spatial organization of the sound field on a wider extent than the set of sampling points.

Current sound field analysis methods can be divided into two categories: the ones based on Kirchhoff's integral equation, which are used mainly in acoustical holographic applications, such as Nearfield Acoustical Holography (NAH) [9] and Wave Field Synthesis [2] (WFS), and the others based on harmonic decompositions like Helmholtz equation least-squares method (HELS) [1] used to study the vibrations of structures, and High Order Ambisonics (HOA) [3]. Some correspondences have already been established between these two kinds of models by Daniel and *al*. [3], in which WFS and HOA systems are compared. This article presents a global synthesis of the different sound field models belonging to these two categories. In particular, it is shown that all harmonic representations, such as plane waves, cylindrical harmonics, and spherical harmonics ones are equivalent from a theoretical point of view, and links are established between harmonic descriptions and the integral representation. For this purpose, ideal conditions are supposed concerning the knowledge of the sound field being analyzed: it is known continuously in the time-space domain.

This paper first recalls general results concerning solutions of the wave equation in section 2. In section 3, harmonic descriptions of the sound field are reviewed: plane waves, cylindrical harmonics, and spherical harmonics. These descriptions rely on the theory of generalized Fourier transforms. In section 4, the holographic approach is investigated, leading to the Kirchhoff's integral equation model. The equivalence between all harmonic decompositions is demonstrated in section 5, and some links are established between harmonic and integral representations. In section 6, we use several analysis methods presented in the previous sections to study the case of truncated plane waves inside a sphere of radius R. Then, the assets and drawbacks of these representations are summarized. Finally, some perspectives are indicated in section 7.

2 Physical background

The three usual coordinate systems, cartesian, cylindrical, and spherical will be used throughout this article. The notations used for a given vector $\mathbf{k}$ are summarized in figure 1.

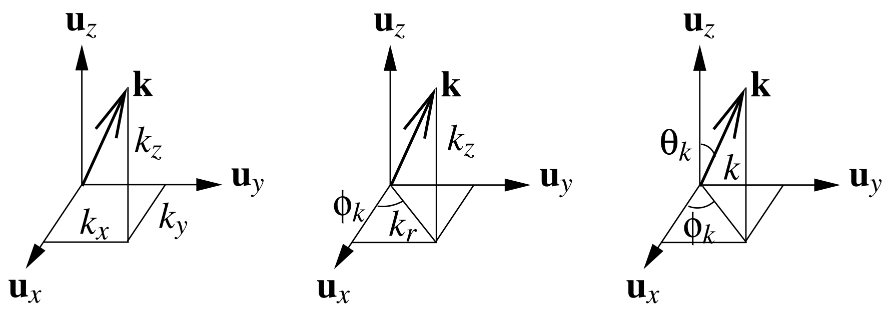

Figure 1: Coordinate systems: cartesian on the left, cylindrical in the middle, and spherical on the right

2.1 The wave equation

The ear is sensitive to the variations of the acoustic pressure $p(\mathbf{r},t)$, where $\mathbf{r}$ and t respectively indicate the space variable (tri-dimensional) and the time variable. This observation justifies the relevance of the acoustic pressure field for sound field analysis. The evolution of the acoustic pressure p is governed by the three-dimensional wave equation [10]:

$$\nabla^2 p(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2} = -q(\mathbf{r},t) \tag{1}$$

where ∇^2 denotes the three-dimensional Laplacian operator, c indicates the sound velocity in the medium (340 m.s^{-1} in the air), and q is a source term which unit is kg.m^{-3}.s^{-2}.

This equation belongs to the category of second order hyperbolic partial differential equations. Additional conditions are required to ensure the existence and uniqueness of the solution [11]. These are the initial conditions, which have to be of *Cauchy* type —the pressure and its first time derivative are known in all the domain of resolution $\mathcal{V}$ at the initial time t_i— and the boundary conditions, which are either of *Dirichlet*, *Neumann*, or *Robin* types —the pressure, its normal derivative, or a linear combination of both is known on the boundary $\mathcal{S}$ delimiting $\mathcal{V}$ at any time instant t.

The inhomogeneous problem stated by equation (1) and by the set of initial and boundary conditions is usually not solved directly. Two different strategies, using elementary solutions, are used for its indirect resolution. The first one uses solutions of the homogeneous problem, reviewed at paragraph 2.2, as a starting point, and extends this set of solutions to deal with inhomogeneous problems. This strategy will be developed in section 3, concerning the analysis of sound fields based on generalized Fourier transforms. The second one is comes from Green's function theory and uses solutions to elementary inhomogeneous problems in conjunction with the superposition principle to deal with the global inhomogeneous problem. This approach will be investigated in section 4.

2.2 Elementary solutions to the homogeneous problem

In this paragraph, several elementary solutions to the homogeneous problem are reviewed. The general principle is to search solutions of the wave equation (1), with a null source term, which are separable in a given coordinate system. In the following, the cartesian, cylindrical, and spherical coordinate systems are considered, but any other coordinate system, such as ellipsoid, or prolate spheroidal could provide other forms of solutions.

2.2.1 Cartesian coordinate system

The solution is assumed to be separable in the cartesian coordinate system, that is:

$$p(\mathbf{r},t) = X(x)Y(y)Z(z)T(t) \tag{2}$$

Introducing this form of solution into (1), with $q(\mathbf{r},t) = 0$, leads to the system of ordinary differential equations [12]:

$$\left\{ \begin{array}{rcl} \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} & = & -k_x^2 \\ \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} & = & -k_y^2 \\ \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} & = & -k_z^2 \\ \frac{1}{T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} & = & -\omega^2 \end{array} \right. \tag{3}$$

The solutions of this system are the well-known plane waves (4), satisfying the dispersion relationship (5):

$$\Psi_{k_x,\,k_y,\,k_z,\,\omega}(x,y,z,t) = e^{\mathrm{i}k_x x + k_y y + k_z z + \omega t} = e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)} \tag{4}$$

$$\text{with } |\mathbf{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2} = k = \frac{\omega}{c} \tag{5}$$

2.2.2 Cylindrical coordinate system

Solutions of the homogeneous wave equation separable in the cylindrical coordinate system are of the form:

$$p(\mathbf{r},t) = R(r)\Phi(\phi)Z(z)T(t) \tag{6}$$

Introducing this prototype of solution into equation (1) yields the following system of ordinary differential equations [12] after some manipulations:

$$\left\{ \begin{array}{rcl} \frac{1}{T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} & = & -\omega^2 \\ \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} & = & -k_z^2 \\ \frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} & = & -l^2 \\ \frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \left(k_r^2 - \frac{l^2}{r^2}\right)R & = & 0 \end{array} \right. \tag{7}$$

Granted that Φ must be a $2\pi-$periodic function, l must be an integer. The solutions of the last equation of the system are the Bessel functions $\mathrm{J}_l(k_r r)$ and $\mathrm{N}_l(k_r r)$ of first and second kind. The Bessel functions of the first kind are continuous at $r = 0$, while the Bessel functions of the second kind are discontinuous at $r = 0$. The general solution is thus a linear combination of (8) and (9), both of these elementary solutions satisfying the dispersion relationship (10):

$$\begin{aligned}
\Psi_{k_r,l,k_z,\omega}(r,\phi,z,t) &= \mathrm{J}_l(k_r r)\, e^{\mathrm{i}l\phi} e^{\mathrm{i}(k_z z+\omega t)} && (8)\\
\Upsilon_{k_r,l,k_z,\omega}(r,\phi,z,t) &= \mathrm{N}_l(k_r r)\, e^{\mathrm{i}l\phi} e^{\mathrm{i}(k_z z+\omega t)} && (9)\\
k_r^2+k_z^2 &= \frac{\omega^2}{c^2} && (10)
\end{aligned}$$

2.2.3 Spherical coordinate system

In this paragraph, the solution is assumed to be separable in the spherical coordinate system:

$$p(\mathbf{r},t) = R(r)\Phi(\phi)\Theta(\theta)T(t) \tag{11}$$

Introducing this form of solution into (1), yields the following system of ordinary differential equations [12] after some manipulations:

$$\left\{\begin{aligned}
\tfrac{1}{T}\tfrac{\mathrm{d}^2T}{\mathrm{d}t^2} &= -\omega^2\\
\tfrac{1}{\Phi}\tfrac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} &= -m^2\\
\tfrac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}+\tfrac{\cos\theta}{\sin\theta}\tfrac{\mathrm{d}\Theta}{\mathrm{d}\theta}+\left(l(l+1)-\tfrac{m}{\sin^2\theta}\right)\Theta &= 0\\
\tfrac{\mathrm{d}^2R}{\mathrm{d}r^2}+\tfrac{2}{r}\tfrac{\mathrm{d}R}{\mathrm{d}r}+\left(k^2-\tfrac{l(l+1)}{r^2}\right)R &= 0
\end{aligned}\right. \tag{12}$$

In these equations, l and m are integers. The solutions of the third equation are the associated *Legendre* functions $\mathrm{P}_l^m(\cos\theta)$, nulled for $|m|>l$. The solutions of the last equation of the system are the spherical Bessel functions $\mathrm{j}_l(kr)$ and $\mathrm{n}_l(kr)$ of first and second kind. The spherical Bessel functions of the first kind are continuous at $r=0$, while the spherical Bessel functions of the second kind are discontinuous at $r=0$. The general solution is thus a linear combination of (13) and (14), both of these elementary solutions satisfying the dispersion relationship (15):

$$\begin{aligned}
\Psi_{k,l,m,\omega}(r,\phi,\theta,t) &= \mathrm{j}_l(kr)\,\mathrm{Y}_l^m(\phi,\theta)\, e^{\mathrm{i}\omega t} && (13)\\
\Upsilon_{k,l,m,\omega}(r,\phi,\theta,t) &= \mathrm{n}_l(kr)\,\mathrm{Y}_l^m(\phi,\theta)\, e^{\mathrm{i}\omega t} && (14)\\
k^2 &= \frac{\omega^2}{c^2} && (15)
\end{aligned}$$

where the $\mathrm{Y}_l^m(\phi,\theta)$ are more usually known as spherical harmonics, whose expression is given by the relation:

$$\mathrm{Y}_l^m(\phi,\theta) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\mathrm{P}_l^m(\cos\theta)\, e^{\mathrm{i}m\phi} \tag{16}$$

2.3 Solution to the elementary inhomogeneous problem

Green's function is the solution to the elementary inhomogeneous problem, in infinite domain, stated by the set of equations:

$$\begin{cases} \nabla^2 \mathrm{G}(\mathbf{r},t|\mathbf{r}_0,t_0) - \dfrac{1}{c^2}\dfrac{\partial^2 \mathrm{G}(\mathbf{r},t|\mathbf{r}_0,t_0)}{\partial t^2} \\ \qquad\qquad = -\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0) \\ \lim\limits_{r\to\infty} \mathrm{G}(\mathbf{r},t|\mathbf{r}_0,t_0) = 0 \\ \mathrm{G}(\mathbf{r},t|\mathbf{r}_0,t_0) = 0 \text{ and } \dfrac{\partial \mathrm{G}(\mathbf{r},t|\mathbf{r}_0,t_0)}{\partial t} = 0 \text{ if } t < t_0 \end{cases} \tag{17}$$

Thus, Green's function is the solution to the homogeneous problem everywhere except at one point located by $\mathbf{r}_0$, at the time instant t_0 [11]. The inhomogeneity $\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0)$ is not a valid acoustic source term, because its unit is in m^{-3}.s^{-1}, instead of kg.m^{-3}.s^{-2}, but Green's function is nevertheless a powerful theoretical tool which will be used to deal with the global inhomogeneous problem at section 4.

A quite complicated development, available in Duffy [13], leads to the well-known result for Green's function in infinite domain:

$$\mathrm{G}(\mathbf{r},t|\mathbf{r}_0,t_0) = \frac{\delta\left(t - t_0 - \frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}_0|} \tag{18}$$

The Dirac delta function is emitted from point $\mathbf{r}_0$ at time instant t_0 and propagates away from the inhomogeneity at the velocity c. It is also noticeable that the acoustic pressure amplitude decreases in $1/|\mathbf{r}-\mathbf{r}_0|$.

3 Sound field analysis based on generalized Fourier transforms

This section is divided into three parts, which are focused on the plane wave, the cylindrical harmonic and the spherical harmonic decompositions. Although based on mathematical results known for several decades now, this synthesis of the generalized harmonic models in one common development is original and quite new.

In section 2.2, several elementary solutions to the homogeneous problem were reviewed. In this section, these sets of solutions will be extended to deal with the inhomogeneous problem, which is the most general one. This strategy naturally falls in the framework of generalized Fourier transforms. In particular, we will see that the analysis tool associated to the plane waves (4) is the usual multidimensional Fourier transform, and that the analysis tools associated to the cylindrical harmonics (8) and spherical harmonics (13) are generalized Fourier transforms.

In a general manner, generalized Fourier transforms enable the expansion of a given sound field —in fact, the acoustical nature of the field is not of major importance, it could also apply to an electromagnetic field, or any other field— in a set of functions, satisfying the two following properties:

- This set of functions forms an orthogonal set with respect to the inner product:

$$< f|g >= \iiiint_{(\mathbf{r},t)\in\mathbb{R}^4} f(\mathbf{r},t)\,\overline{g}(\mathbf{r},t)\;\mathrm{d}^3\mathbf{r}\,\mathrm{d}t \tag{19}$$

  where $\overline{g}$ denotes the complex conjugate of g.

- This set of functions is *complete*, which means that a "well-behaved"[1] sound field can be perfectly recovered from the knowledge of its harmonic representation.

3.1 Plane wave expansion

The set of solutions to the homogeneous wave equation, separable in the cartesian coordinate system are the plane waves (4) satisfying the dispersion relationship (5). To deal with the inhomogeneous problem, the dispersion relationship is no longer assumed, and we consider the set of plane waves for every $(\mathbf{k}, \omega) \in \mathbb{R}^4$. This extended set of functions is orthogonal with respect to the inner product (19). Indeed:

$$\begin{aligned} &< \Psi_{k_{x_1},\, k_{y_1},\, k_{z_1}, \omega_1} | \Psi_{k_{x_2},\, k_{y_2},\, k_{z_2}, \omega_2} > \\ &= \int_{x=-\infty}^{+\infty} e^{\mathrm{i}\left(k_{x_1}-k_{x_2}\right)x}\, \mathrm{d}x \int_{y=-\infty}^{+\infty} e^{\mathrm{i}\left(k_{y_1}-k_{y_2}\right)y}\, \mathrm{d}y \\ &\quad \int_{z=-\infty}^{+\infty} e^{\mathrm{i}\left(k_{z_1}-k_{z_2}\right)z}\, \mathrm{d}z \int_{t=-\infty}^{+\infty} e^{\mathrm{i}(\omega_1-\omega_2)t}\, \mathrm{d}t \\ &= (2\pi)^4\, \delta(k_{x_1}-k_{x_2})\, \delta(k_{y_1}-k_{y_2})\, \delta(k_{z_1}-k_{z_2})\, \delta(\omega_1-\omega_2) \end{aligned} \tag{20}$$

This result comes from the famous identity (63) satisfied by Dirac's delta function. Thus, any sound field can be projected on the set of the plane waves. The analysis operator associated to this operation is the usual multidimensional Fourier transform:

$$\begin{aligned} \mathcal{P}\{p\}(k_x, k_y, k_z, \omega) &= < p | \Psi_{k_x, k_y, k_z, \omega} > \\ &= \iiiint_{(\mathbf{r},t)\in\mathbb{R}^4} p(\mathbf{r}, t)\, e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)} \mathrm{d}^3\mathbf{r}\, \mathrm{d}t \end{aligned} \tag{21}$$

Moreover, the set of plane waves $(\Psi_{\mathbf{k},\omega})$ with $(\mathbf{k}, \omega) \in \mathbb{R}^4$ is complete, because of the Fourier transform theorem:

$$\begin{aligned} p(\mathbf{r}, t) = \frac{1}{(2\pi)^4} &\iiiint_{(\mathbf{k},\omega)\in\mathbb{R}^4} e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)} \\ &\left[\iiiint_{(\mathbf{r_1},t_1)\in\mathbb{R}^4} p(\mathbf{r_1}, t_1)\, e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r_1}+\omega t_1)} \mathrm{d}^3\mathbf{r_1}\, \mathrm{d}t_1\right] \mathrm{d}^3\mathbf{k}\, \mathrm{d}\omega \end{aligned} \tag{22}$$

The Fourier transform theorem implies that a sound field can be perfectly recovered by the knowledge of its Fourier transform. The associated synthesis operator is given by the next equation, which is the usual multidimensional inverse Fourier transform:

$$p(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \iiiint_{\mathbf{r},t\in\mathbb{R}^4} \mathcal{P}\{p\}(k_x, k_y, k_z, \omega)\, e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)} \mathrm{d}^3\mathbf{k}\, \mathrm{d}\omega \tag{23}$$

3.2 Cylindrical harmonics expansion

Cylindrical harmonics (8) satisfying the dispersion relationship (10) are also harmonic solutions to the homogeneous wave equation. In the same manner as in the previous paragraph, the dispersion relationship is no longer assumed to deal with the inhomogeneous problem, and the set of cylindrical harmonics with $[k_r, l, k_z, \omega] \in \mathbb{R}^+ \times \mathbb{Z} \times \mathbb{R}^2$

[1]theoretical study of the meaning of "well-behaved" is beyond the scope of this article. But, for physical fields, it is usually the case.

is used instead. This extended set of functions is orthogonal with respect to the inner product (19). Indeed, using a volume element described in cylindrical coordinates, $d^3\mathbf{r} = r\ dr\ d\phi\ dz$, the inner product of two cylindrical harmonics is given by the next equation:

$$
\begin{aligned}
&< \Psi_{k_{r_1},\, l_1,\, k_{z_1},\, \omega_1} | \Psi_{k_{r_2},\, l_2,\, k_{z_2},\, \omega_2} > \\
&= \int_{r=0}^{+\infty} \mathrm{J}_{l_1}(k_{r_1} r)\, \mathrm{J}_{l_2}(k_{r_2} r)\, r\ \mathrm{d}r \int_{\phi=0}^{2\pi} e^{\mathrm{i}(l_1-l_2)\phi}\ \mathrm{d}\phi \\
&\int_{z=-\infty}^{+\infty} e^{\mathrm{i}\left(k_{z_1}-k_{z_2}\right)z}\ \mathrm{d}z \int_{t=-\infty}^{+\infty} e^{\mathrm{i}(\omega_1-\omega_2)t}\ \mathrm{d}t \\
&= (2\pi)^3 \frac{\delta(k_{r_1}-k_{r_2})}{k_{r_1}} \delta_{l_1 l_2}\, \delta(k_{z_1}-k_{z_2})\, \delta(\omega_1-\omega_2)
\end{aligned}
\tag{24}
$$

The integral for the azimuth variable ϕ is an elementary result of decomposition into Fourier series, the integral for the variables z and t use the identity (63), and the radial integral uses the *closure relation* of Bessel functions of the first kind (64).

Any sound field can be projected on the set of the cylindrical harmonics. The analysis operator associated to this operation is a generalized Fourier transform, which will be referred to as the cylindrical harmonic transform hereafter:

$$
\begin{aligned}
\mathcal{C}\{p\}(k_r, l, k_z, \omega) \quad &= \quad < p | \Psi_{k_r,\, l,\, k_z,\, \omega} > \\
= \int_{r=0}^{+\infty} \int_{\phi=0}^{2\pi} \iint_{(z,t)\in\mathbb{R}^2} & p(\mathbf{r},t)\, \mathrm{J}_l(k_r r)\, e^{-\mathrm{i}(l\phi + k_z z + \omega t)} r\ \mathrm{d}r\ \mathrm{d}\phi\ \mathrm{d}z\ \mathrm{d}t
\end{aligned}
\tag{25}
$$

This cylindrical harmonic transform is the product of an Hankel transform of order l for the radial variable r, a decomposition into Fourier series for the azimuthal variable ϕ, and two Fourier transforms for the space variable z and the time variable t.

Moreover, the set of cylindrical harmonics $\left(\Psi_{k_r,l,k_z,\omega}\right)$ with $[k_r, l, k_z, \omega] \in \mathbb{R}^+ \times \mathbb{Z} \times \mathbb{R}^2$ is complete because of the following generalized Fourier transform theorem:

$$
\begin{aligned}
p(\mathbf{r},t) = \frac{1}{(2\pi)^3} \sum_{l=-\infty}^{+\infty} & \iiint_{(k_r,k_z,\omega)\in\mathbb{R}^+\times\mathbb{R}^2} \\
\Bigg[\int_{r_1=0}^{+\infty} \int_{\phi_1=0}^{2\pi} & \int_{z_1=-\infty}^{+\infty} \int_{t_1=-\infty}^{+\infty} p(\mathbf{r_1}, t_1) \\
& \mathrm{J}_l(k_r r_1)\, e^{-\mathrm{i}(l\phi_1 + k_z z_1 + \omega t_1)} r_1\ \mathrm{d}r_1 \mathrm{d}\phi_1\ \mathrm{d}z_1\ \mathrm{d}t_1 \Bigg]. \\
& \mathrm{J}_l(k_r r)\, e^{\mathrm{i}(l\phi + k_z z + \omega t)} k_r\ \mathrm{d}k_r\ \mathrm{d}k_z\ \mathrm{d}\omega
\end{aligned}
\tag{26}
$$

The key-point of the demonstration lies in the inversion of the two integrals. Indeed:

$$
\begin{aligned}
\sum_{l=-\infty}^{+\infty} e^{\mathrm{i}l(\phi-\phi_1)} & \int_{k_r=0}^{+\infty} \mathrm{J}_l(k_r r)\, \mathrm{J}_l(k_r r_1)\, k_r\ \mathrm{d}k_r. \\
\int_{k_z=-\infty}^{+\infty} & e^{\mathrm{i}k_z(z-z_1)} \mathrm{d}k_z \int_{t=-\infty}^{+\infty} e^{i\omega(t-t_1)} \mathrm{d}\omega = \\
& (2\pi)^3\, \delta(\phi-\phi_1)\, \frac{\delta(r-r_1)}{r_1}\, \delta(z-z_1)\, \delta(t-t_1)
\end{aligned}
\tag{27}
$$

Inserting this result in the integral over $(\mathbf{r_1}, t_1)$ proves the above theorem (26).

This generalized Fourier transform theorem implies that a sound field can be perfectly recovered knowing its cylindrical harmonic transform. The corresponding synthesis operator associated to this operation is the inverse cylindrical harmonic transform:

$$p(\mathbf{r},t) = \frac{1}{(2\pi)^3} \sum_{l=-\infty}^{+\infty} \int_{k_r=0}^{+\infty} \iint_{(k_z,\omega)\in\mathbb{R}^2} \mathcal{C}\{p\}(k_r,l,k_z,\omega)\,. \\ \mathrm{J}_l(k_r r)\, e^{\mathrm{i}(l\phi+k_z z+\omega t)} k_r\, \mathrm{d}k_r\, \mathrm{d}k_z\, \mathrm{d}\omega \tag{28}$$

3.3 Spherical harmonics expansion

To deal with the inhomogeneous problem, spherical harmonics (13) with $[k,l,m,\omega] \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{Z} \times \mathbb{R}$ are used instead of the limited set of spherical harmonics satisfying the dispersion relationship (15). This set of functions is orthogonal with respect to the inner product defined by the relationship (19). Using an elementary volume described in spherical coordinates $\mathrm{d}^3\mathbf{r} = r^2\ \mathrm{d}r\ \mathrm{d}\phi\ \sin\theta\ \mathrm{d}\theta$, the inner product of two spherical harmonics is given by the next equation:

$$\begin{aligned} &< \Psi_{k_1,\, l_1,\, m_1,\, \omega_1} | \Psi_{k_2,\, l_2,\, m_2,\, \omega_2} > = \\ &\int_{r=0}^{+\infty} \mathrm{j}_{l_1}(k_1 r)\, \mathrm{j}_{l_2}(k_2 r)\, r^2\, \mathrm{d}r \int_{t=-\infty}^{+\infty} e^{\mathrm{i}(\omega_1-\omega_2)t}\ \mathrm{d}t\ . \\ &\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \mathrm{Y}_{l_1}^{m_1}(\phi,\theta)\, \overline{\mathrm{Y}_{l_2}^{m_2}}(\phi,\theta)\ \mathrm{d}\phi\ \sin\theta\ \mathrm{d}\theta = \\ &\frac{4}{k_1 k_2} \delta(k_1-k_2)\, \delta_{l_1 l_2} \delta_{m_1 m_2} \delta(\omega_1-\omega_2) \end{aligned} \tag{29}$$

For this equation, the radial integral is given by the *closure relation* for the spherical Bessel functions (66), the time integral is given by (63), and the angular integral is given by the orthogonality property (67) of the classical spherical harmonics $\mathrm{Y}_l^m(\phi,\theta)$.

Any sound field can be projected on the set of the spherical harmonics. The analysis operator associated to this operation is a generalized Fourier transform, which will be referred to as the spherical harmonic transform hereafter:

$$\begin{aligned} \mathcal{S}\{p\}(k,l,m,\omega) &= < p | \Psi_{k,\, l,\, m,\, \omega} > \\ &= \int_{r=0}^{+\infty} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{t=-\infty}^{+\infty} p(\mathbf{r},t) \\ &\quad \mathrm{j}_l(kr)\, \overline{\mathrm{Y}_l^m}(\phi,\theta)\, e^{-\mathrm{i}\omega t} r^2\ \mathrm{d}r\ \mathrm{d}\phi\ \sin\theta\ \mathrm{d}\theta\ \mathrm{d}t \end{aligned} \tag{30}$$

This spherical harmonic transform is the product of a spherical Hankel transform of order l for the radial variable r, a decomposition into classical spherical harmonics for the solid angle Ω, and a Fourier transform for the time variable t.

Moreover, the set of spherical harmonics $(\Psi_{k,l,m,\omega})$ with $[k,l,m,\omega] \in \mathbb{R}^+ \times \mathbb{N} \times \mathbb{Z} \times$

$\mathbb{R}$ is complete because of the following generalized Fourier transform theorem:

$$
\begin{aligned}
p(\mathbf{r},t) = \frac{1}{4}\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} & \iint_{(k,\omega)\in\mathbb{R}^+\times\mathbb{R}} \\
& \left[\int_{r_1=0}^{+\infty}\int_{\phi_1=0}^{2\pi}\int_{\theta_1=0}^{\pi}\int_{t_1=-\infty}^{+\infty} p(\mathbf{r_1},t_1)\right. \\
& \left. \mathrm{j}_l(kr_1)\,\overline{\mathrm{Y}_l^m}(\phi_1,\theta_1)\,e^{-\mathrm{i}\omega t_1} r_1^2\,\mathrm{d}r_1\mathrm{d}\phi_1\ \sin\theta_1\mathrm{d}\theta_1\ \mathrm{d}t_1\right]. \\
& \mathrm{j}_l(kr)\,\mathrm{Y}_l^m(\phi,\theta)\,e^{\mathrm{i}\omega t}k^2\,\mathrm{d}k\,\mathrm{d}\omega
\end{aligned}
\tag{31}
$$

The key-point of the demonstration lies in the inversion of the two integrals. Indeed:

$$
\begin{aligned}
& \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\mathrm{Y}_l^m(\phi,\theta)\,\overline{\mathrm{Y}_l^m}(\phi_1,\theta_1)\int_{k=0}^{+\infty}\mathrm{j}_l(kr)\mathrm{j}_l(kr_1)\,k^2\,\mathrm{d}k\,. \\
& \int_{t=-\infty}^{+\infty} e^{i\omega(t-t_1)}\mathrm{d}\omega = 4\frac{\delta(r-r_1)}{r_1^2} \\
& \delta(\phi-\phi_1)\,\delta(\cos\theta-\cos\theta_1)\,\delta(t-t_1)
\end{aligned}
\tag{32}
$$

Inserting this result in the integral over $(\mathbf{r_1},t_1)$ proves the above theorem (31).

This generalized Fourier transform theorem implies that a sound field can be perfectly recovered knowing its spherical harmonic transform. The corresponding synthesis operator associated to this operation is the inverse spherical harmonic transform:

$$
\begin{aligned}
p(\mathbf{r},t) = \frac{1}{4}\sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\int_{k=0}^{+\infty}\int_{\omega=-\infty}^{+\infty} & \mathcal{S}\{p\}(k,l,m,\omega)\,. \\
& \mathrm{j}_l(kr)\,\mathrm{Y}_l^m(\phi,\theta)\,e^{\mathrm{i}\omega t}k^2\,\mathrm{d}k\,\mathrm{d}\omega
\end{aligned}
\tag{33}
$$

3.4 Conclusion

The analysis operators and synthesis operators of several harmonic descriptions have been given throughout this section: (21) and (23) for the plane waves, (25) and (28) for the cylindrical harmonics, (30) and (33) for the spherical harmonics. They are based on generalized Fourier transforms which are invertible for “well-behaved” sound fields, so that these harmonic decompositions are suitable models for sound field description. Examples of use of these generalized Fourier transforms to deal with inhomogeneous problems will be given at section 5.2.

4 Sound field analysis based on Kirchhoff’s integral

4.1 Kirchhoff’s integral equation

While in the previous section the solution of the general inhomogeneous problem stated by equation (1) with initial and boundary conditions was expressed as a superposition of harmonic solutions, another expression of the sound field could be given using Green’s function theory. This leads to an integral expression of the sound field, known as Kirchhoff's integral [11], whose most general form is:

$$\begin{aligned}
&\int_{t_0=t_i}^{t^+} \oiint_{\mathbf{r}_0 \in \mathcal{S}} \left[\mathrm{G}\left(\mathbf{r},t|\mathbf{r}_0,t_0\right) \nabla_0 p\left(\mathbf{r}_0,t_0\right) \right. \\
&\qquad\qquad \left. - p\left(\mathbf{r}_0,t_0\right) \nabla_0 \mathrm{G}\left(\mathbf{r},t|\mathbf{r}_0,t_0\right)\right] \cdot \mathrm{d}\mathbf{S}_0 \, \mathrm{d}t_0 \\
&+ \frac{1}{c^2} \iiint_{\mathbf{r}_0 \in \mathcal{V}} \left[\mathrm{G}\left(\mathbf{r},t|\mathbf{r}_0,t_i\right) \frac{\partial p\left(\mathbf{r}_0,t_i\right)}{\partial t_0} \right. \\
&\qquad\qquad \left. - p\left(\mathbf{r}_0,t_i\right) \frac{\partial \mathrm{G}\left(\mathbf{r},t|\mathbf{r}_0,t_i\right)}{\partial t_0} \right] \mathrm{d}^3\mathbf{r}_0 \\
&+ \int_{t_0=t_i}^{t^+} \iiint_{\mathbf{r}_0 \in \mathcal{V}} q\left(\mathbf{r}_0,t_0\right) \mathrm{G}\left(\mathbf{r},t|\mathbf{r}_0,t_0\right) \mathrm{d}^3\mathbf{r}_0 \, \mathrm{d}t_0 \\
&\qquad\qquad = \begin{cases} p\left(\mathbf{r},t\right) & \text{if } \mathbf{r} \in \mathcal{V} \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{34}$$

where t_i is the initial time, index 0 in ∇_0 indicates that the gradient is with respect to the variable $\mathbf{r}_0$, $\mathcal{V}$ and $\mathcal{S}$ are the domain of resolution and the surface delimiting it, and $\mathrm{d}\mathbf{S}_0$, which is normal to the surface, points outward $\mathcal{V}$.

The first term of the left member of the above equation (34) is the contribution from inhomogeneous —non null— boundary conditions, the second term is the contribution from inhomogeneous initial conditions, and the third term is the contribution from the non null source term inside the domain of resolution $\mathcal{V}$.

Several simplifications of this integral equation occur according to the initial and boundary conditions satisfied by Green's function. For instance, if there are no sources inside the domain of resolution, if initial conditions are null and if Green's function is imposed to satisfy homogeneous Dirichlet boundary conditions, then the equation (34) comes down to:

$$\begin{aligned}
&\int_{t_0=t_i}^{t^+} \oiint_{\mathbf{r}_0 \in \mathcal{S}} p\left(\mathbf{r}_0,t_0\right) \nabla_0 \mathrm{G}\left(\mathbf{r},t|\mathbf{r}_0,t_0\right) \cdot \mathrm{d}\mathbf{S}_0 \, \mathrm{d}t_0 \\
&\qquad\qquad = \begin{cases} -p\left(\mathbf{r},t\right) & \text{if } \mathbf{r} \in \mathcal{V} \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{35}$$

This formulation of Kirchhoff's integral equation is only of theoretical interest, because there are only very few cases where an analytical Green's function satisfying the required initial and boundary conditions is available. Nevertheless, it implies that the reconstruction of a sound field is possible exclusively from the knowledge of the acoustic pressure profile on the boundary.

4.2 Huygens' principle

When an analytical form of the Green's function leading to the integral equation (35) is unknown, using Green's function in infinite domain (18) is usually preferred. If initial conditions are null, and if there are no sources inside the domain of resolution, this leads to Kirchhoff's formulation of Huygens' principle:

$$\begin{aligned}
&\frac{1}{4\pi}\int_{t_0=t_i}^{t^+}\oiint_{\mathbf{r}_0\in\mathcal{S}}\left[\frac{\delta\left(t-t_0-\frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|}\nabla_0 p\left(\mathbf{r}_0,t_0\right)\right.\\
&\qquad\left.-p\left(\mathbf{r}_0,t_0\right)\nabla_0\frac{\delta\left(t-t_0-\frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0|}\right]\cdot\mathrm{d}\mathbf{S}_0\,\mathrm{d}t_0\\
&=\begin{cases} p\left(\mathbf{r},t\right) & \text{if } \mathbf{r}\in\mathcal{V}\\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{36}$$

By analogy with section 3, equation (36) acts as a synthesis operator because it enables the recovery of a given sound field from the knowledge of the profiles of the acoustic pressure and its normal derivative with respect to the surface element $\mathrm{d}\mathbf{S}_0$. The analysis operator associated to this integral representation has to provide the knowledge of these profiles, which is formally given by the set of inner products:

$$\begin{cases} < p|\delta_{\mathbf{r}_0,t_0} >= p\left(\mathbf{r}_0,t_0\right)\\ < p|\nabla_0\delta_{\mathbf{r}_0,t_0} >= -\nabla_0 p\left(\mathbf{r}_0,t_0\right)\end{cases} \tag{37}$$

The second equation is obtained using property (68) relative to the first derivative of Dirac's delta function.

5 Links between the different sound field models

The question of the equivalence of the models naturally arises at this point of the study. In fact, if there is such an equivalence, one can choose the best-suited model to the problem under consideration. In this section, we will demonstrate the exact equivalence between the harmonic decompositions presented in section 3, and we will show interesting correspondences between the models introduced in sections 3 and 4. This latter case is particularly interesting because it includes a correspondence between High Order Ambisonics [3] and Wave Field Synthesis [15] approaches.

5.1 Equivalence between harmonic decompositions

In section 3, it has been shown that all harmonic representations were suitable for sound field description because these harmonics form complete sets of functions. Thus, we implicitly know that all these reviewed harmonic descriptions are equivalent. In this part, we are interested in giving analytical forms of some basis change operators. They enable the conversion of a sound field harmonic representation into another one.

5.1.1 Plane wave to cylindrical harmonic basis change operator

Starting from the plane wave harmonic representation $\mathcal{P}\{p\}(k_x,k_y,k_z,\omega)$, one can obtain its cylindrical harmonic representation $\mathcal{C}\{p\}(K_r,L,K_z,\Omega)$ by applying the analysis operator for cylindrical harmonics (25), but replacing $p(\mathbf{r},t)$ by equation (23). This strategy will also be applied for all other conversions. Using the following notations (see figure 1), $\mathbf{k_r}=k_x\,\mathbf{u_x}+k_y\,\mathbf{u_y}$, $k_r=|\mathbf{k_r}|$, $\phi_k=(\mathbf{u_x},\mathbf{k_r})$, $\mathbf{r}=x\,\mathbf{u_x}+y\,\mathbf{u_y}$, $r=|\mathbf{r}|$, and $\phi_r=(\mathbf{u_x},\mathbf{r})$, this yields:

$$\begin{array}{l} \mathcal{C}\left\{p\right\}(K_r,\ L,\ K_z,\ \Omega)=\iiiint_{(\mathbf{r},t)\in\mathbb{R}^4}\dfrac{1}{(2\pi)^4}\cdot \\ \left[\iiiint_{(\mathbf{k},\omega)\in\mathbb{R}^4}\mathcal{P}\left\{p\right\}(k_x,k_y,k_z,\omega)\,e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)}\mathrm{d}^3\mathbf{k}\mathrm{d}\omega\right]\cdot \\ \mathrm{J}_L(K_r r)\,e^{-\mathrm{i}(L\phi_r+K_z+\Omega t)}\ \mathrm{d}^3\mathbf{r}\ \mathrm{d}t \end{array}$$

Inverting the two integrals, one obtains:

$$\begin{array}{l} \mathcal{C}\left\{p\right\}(K_r,\ L,\ K_z,\ \Omega)=\dfrac{1}{(2\pi)^4}\iiiint_{(\mathbf{k},\omega)\in\mathbb{R}^4} \\ \mathcal{P}\left\{p\right\}(k_x,k_y,k_z,\omega) < \Psi^{\mathrm{pw}}_{k_x,\ k_y,\ k_z,\ \omega}|\Psi^{\mathrm{ch}}_{K_r,\ L,\ K_z,\ \Omega} > \mathrm{d}^3\mathbf{k}\mathrm{d}\omega \end{array} \tag{38}$$

where pw and ch stand for plane wave and cylindrical harmonic. The inner product is now developed using the Jacobi-Anger identity (69):

$$\begin{array}{l} \iiiint_{(\mathbf{r},t)\in\mathbb{R}^4} e^{\mathrm{i}(k_r r\cos(\phi_r-\phi_k)+k_z z+\omega t)}\mathrm{J}_L\left(K_r r\right)e^{-\mathrm{i}(L\phi_r+K_z z+\Omega t)}\ \mathrm{d}^3\mathbf{r}\ \mathrm{d}t \\ =\displaystyle\sum_{l=-\infty}^{+\infty}\mathrm{i}^l\int_{r=0}^{+\infty}\mathrm{J}_l\left(k_r r\right)\mathrm{J}_L\left(K_r r\right)r\ \mathrm{d}r\int_{\phi_r=0}^{2\pi}e^{-\mathrm{i}l\phi_k}e^{\mathrm{i}(l-L)\phi_r}\ \mathrm{d}\phi_r \\ \quad\cdot\displaystyle\int_{z=-\infty}^{+\infty}e^{\mathrm{i}(k_z-K_z)z}\ \mathrm{d}z\int_{t=-\infty}^{+\infty}e^{i(\omega-\Omega)t}\ \mathrm{d}t \\ =(2\pi)^3\,\mathrm{i}^L e^{-\mathrm{i}L\phi_k}\dfrac{\delta\left(k_r-K_r\right)}{k_r}\delta\left(k_z-K_z\right)\delta\left(\omega-\Omega\right) \end{array} \tag{39}$$

Inserting this last equation back in (38), and using the variables k_r and ϕ_k instead of k_x and k_y in the plane waves model yields:

$$\begin{array}{l} \mathcal{C}\left\{p\right\}(K_r,\ L,\ K_z,\ \Omega)= \\ \dfrac{1}{2\pi}\displaystyle\int_{\phi_k=0}^{2\pi}\mathrm{i}^L\mathcal{P}\left\{p\right\}(K_r,\phi_k,K_z,\Omega)\,e^{-\mathrm{i}L\phi_k}\ \mathrm{d}\phi_k \end{array} \tag{40}$$

Equation (40) enables the direct conversion of a plane wave sound field description into cylindrical harmonics. To interpret this equation, consider the decomposition into Fourier series of the Fourier transform $\mathcal{P}\left\{p\right\}(k_x,k_y,k_z,\omega)$ on the circle defined by the set of equations : $k_x^2+k_y^2=K_r^2$, $k_z=K_z$ and $\omega=\Omega$. Its L^{th} coefficient is equal to $\mathcal{C}\left\{p\right\}(K_r,L,K_z,\Omega)$ except for the scale factor i^L.

5.1.2 Cylindrical harmonic to plane wave basis change operator

The same notations as the previous paragraph still apply here. We are interested in the direct conversion of a cylindrical harmonic sound field description $\mathcal{C}\left\{p\right\}(k_r,l,k_z,\omega)$ into its plane waves one $\mathcal{P}\left\{p\right\}(K_x,K_y,K_z,\Omega)$. Following an analogous reasoning to equation (38), we obtain the intermediate result:

$$\begin{array}{l} \mathcal{P}\left\{p\right\}(K_x,K_y,K_z,\Omega)=\dfrac{1}{(2\pi)^3}\displaystyle\sum_{l=-\infty}^{+\infty}\int_{k_r=0}^{+\infty}\iint_{(k_z,\omega)\in\mathbb{R}^2} \\ \mathcal{C}\left\{p\right\}(k_r,l,k_z,\omega) < \Psi^{\mathrm{ch}}_{k_r,\ l,\ k_z,\ \omega}|\Psi^{\mathrm{pw}}_{K_x,K_y,K_z,\Omega} > k_r\ \mathrm{d}k_r\ \mathrm{d}k_z\ \mathrm{d}\omega \end{array} \tag{41}$$

Using the following notations, $\mathbf{K_r}=K_x\ \mathbf{u_x}+K_y\ \mathbf{u_y}$, $K_r=|\mathbf{K_r}|$, $\phi_K=\underline{(\mathbf{u_x},\mathbf{K_r})}$, updating the notations of equation (39), and using the property $< g|f >=\overline{< f|g >}$, this yields:

$$\mathcal{P}\{p\}(K_x, K_y, K_z, \Omega) = \sum_{l=-\infty}^{+\infty} (-\mathrm{i})^l \, \mathcal{C}\{p\}(K_r, l, K_z, \Omega) \, e^{\mathrm{i}l\phi_K} \tag{42}$$

While the conversion from plane waves to cylindrical harmonics required the decomposition into Fourier series of the Fourier transform taken on a circle, except for the scale factor i^L, the conversion from cylindrical harmonics to plane waves is done by computing the Fourier series associated to the cylindrical harmonics coefficients $\mathcal{C}\{p\}(K_r, l, K_z, \Omega)$, except for the same Hermitian symmetrical scale factor $\overline{\mathrm{i}^l}$. Operators (40) and (42) are perfectly symmetrical in the Fourier sense.

5.1.3 Plane wave to spherical harmonic basis change operator

Applying the spherical harmonic analysis operator (30), but replacing $p(\mathbf{r}, t)$ by the plane wave synthesis operator (23), following an analogous reasoning to equation (38) leads to the intermediate result:

$$\begin{aligned} \mathcal{S}\{p\}(K,\ L,\ M,\ \Omega) = \frac{1}{(2\pi)^4} \iiiint_{(\mathbf{k},\omega)\in\mathbb{R}^4} \\ \mathcal{P}\{p\}(k_x, k_y, k_z, \omega) < \Psi^{\mathrm{pw}}_{k_x,\ k_y,\ k_z,\ \omega} | \Psi^{\mathrm{sh}}_{K,\ L,\ M,\ \Omega} > \mathrm{d}^3\mathbf{k}\ \mathrm{d}\omega \end{aligned} \tag{43}$$

where superscript sh stands for spherical harmonic. The inner product is now developed using the identity (70), analogous to the Jacobi-Anger, but involving spherical Bessel functions:

$$\begin{aligned} &\iiiint_{(\mathbf{r},t)\in\mathbb{R}^4} e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)} \mathrm{j}_L(Kr) \overline{\mathrm{Y}_L^M}(\phi_r, \theta_r) \, e^{-\mathrm{i}\Omega t} \ \mathrm{d}^3\mathbf{r}\ \mathrm{d}t \\ &= 4\pi \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} \mathrm{i}^l \int_{r=0}^{+\infty} \mathrm{j}_l(kr) \mathrm{j}_L(Kr) \, r^2 \ \mathrm{d}r \int_{t=-\infty}^{+\infty} e^{i(\omega-\Omega)t} \ \mathrm{d}t \\ &\cdot \int_{\phi_r=0}^{2\pi} \int_{\theta_r=0}^{\pi} \overline{\mathrm{Y}_l^m}(\phi_k, \theta_k) \mathrm{Y}_l^m(\phi_r, \theta_r) \overline{\mathrm{Y}_L^M}(\phi_r, \theta_r) \ \mathrm{d}\phi_r \sin\theta_r \ \mathrm{d}\theta_r \\ &= 16\pi \ \mathrm{i}^L \overline{\mathrm{Y}_L^M}(\phi_k, \theta_k) \frac{\delta(k-K)}{k^2} \delta(\omega - \Omega) \end{aligned} \tag{44}$$

where the vectors $\mathbf{r}$ and $\mathbf{k}$ are indifferently written in cartesian coordinates (k_x, k_y, k_z) or spherical coordinates (k, ϕ_k, θ_k). Inserting this last equation into (43) gives:

$$\begin{aligned} \mathcal{S}\{p\}(K,\ L,\ M,\ \Omega) = \frac{1}{\pi^3} \int_{\phi_k=0}^{2\pi} \int_{\theta_k=0}^{\pi} \\ \cdot \left[\mathrm{i}^L \mathcal{P}\{p\}(K, \phi_k, \theta_k, \Omega) \, \overline{\mathrm{Y}_L^M}(\phi_k, \theta_k) \right] \mathrm{d}\phi_k \sin\theta_k \ \mathrm{d}\theta_k \end{aligned} \tag{45}$$

To interpret the above equation equation, consider the decomposition into classical spherical harmonics of the Fourier transform $\mathcal{P}\{p\}(k_x, k_y, k_z, \omega)$ evaluated on the sphere defined by the set of equations : $k_x^2 + k_y^2 + k_z^2 = K^2$, and $\omega = \Omega$. Its coefficient indexed by L and M is equal to $\mathcal{S}\{p\}(K, L, M, \Omega)$ except for the scale factor i^L.

5.1.4 Spherical harmonics to plane waves basis change operator

We are interested in the direct conversion of a spherical harmonic sound field description $\mathcal{S}\{p\}(k,l,m,\omega)$ into plane waves $\mathcal{P}\{p\}(K_x,K_y,K_z,\Omega)$. Following an analogous reasoning to equation (38), we obtain the intermediate result:

$$\begin{aligned}\mathcal{P}\{p\}(K_x,K_y,K_z,\Omega) &= \frac{1}{4}\sum_{l=0}^{+\infty}\sum_{m=-l}^{l}\int_{k=0}^{+\infty}\int_{\omega\in\mathbb{R}} \\ \mathcal{S}\{p\}(k,l,m,\omega) &< \Psi^{\mathrm{sh}}_{k,\ l,\ m,\ \omega}|\Psi^{\mathrm{pw}}_{K_x,K_y,K_z,\Omega} > k^2\,\mathrm{d}k\,\mathrm{d}\omega\end{aligned} \tag{46}$$

Using either cartesian coordinates (K_x,K_y,K_z) or spherical coordinates (K,Φ_K,θ_K) for the vector $\mathbf{K}$, updating the notations of equation (44), and using the property $<g|f> = \overline{<f|g>}$, this yields:

$$\begin{aligned}&\mathcal{P}\{p\}(K_x,K_y,K_z,\Omega) = \\ &\qquad 4\pi\sum_{l=-\infty}^{+\infty}\sum_{m=-l}^{l}(-\mathrm{i})^l\,\mathcal{S}\{p\}(K,l,m,\Omega)\,\mathrm{Y}_l^m(\phi_K,\theta_K)\end{aligned} \tag{47}$$

While the conversion from plane waves to spherical harmonics required the decomposition into spherical harmonics of the Fourier transform evaluated on a sphere, except for the scale factor i^L, the conversion from spherical harmonics to plane waves is done by computing the classical spherical harmonics series associated to the spherical harmonics coefficients $\mathcal{S}\{p\}(K,l,m,\Omega)$, except for the same Hermitian symmetrical scale factor $\overline{\mathrm{i}^l}$. Operators (45) and (47) are also perfectly symmetrical in the Fourier sense.

5.1.5 Other basis change operators

The basis change operators between cylindrical harmonics and spherical harmonics remain to be treated. The same approach as the one adopted in the previous paragraphs could be used in order to derive these operators. This would lead to complex formulas, which are not very useful. Indeed, sound field reproduction systems based on High Order Ambisonics [3] rely upon the theory of spherical harmonics, while their restriction to the 2D case rely upon the theory of cylindrical harmonics, where the reference to the variable z is omitted in the previous developments. Thus, the basis change operators between plane waves and cylindrical/spherical harmonics link the HOA with the usual multidimensional Fourier transform. On the other hand, the parallel between cylindrical harmonics and spherical harmonics is complicated and not very useful for practical purposes.

5.2 Correspondences between harmonic decompositions and Kirchhoff's integral

Harmonic decompositions are more general than Kirchhoff's integral, because these sets of functions are complete, as shown by the generalized Fourier transform theorems (22), (26) and (31). Thus, the nature of the 4D signal being analyzed with generalized Fourier transforms does not really matter, as mentioned in section 3. Nevertheless, knowing that these signals are sound fields gives *a priori* information, such as the dispersion relationships (5), (10) and (15) for instance. On the other hand, a sound field described by Kirchhoff's integral always satisfies the wave equation (1).

The space spanned by the sets of harmonic functions is more extended than the space spanned by Kirchhoff's integral. The first part of this development deals with the conversion of an integral sound field representation into an harmonic one. This conversion is not destructive. The second part is concerned by deriving the integral sound field representation from a harmonic one, and the limits of this conversion are pointed out.

5.2.1 Conversion of integral sound field representations into harmonic ones

The generalization of Kirchhoff's form of Huygens' principle (36) makes use of Green's function $G(\mathbf{r},t|\mathbf{r}_0,t_0)$ in infinite domain and its gradient. This representation can be converted into an harmonic one, by using the same strategy as in all paragraphs of the previous section 5.1. For instance, using the analysis operator for plane waves (21), but replacing $p(\mathbf{r},t)$ by its integral representation (36) yields, after the inversion of the two integrals:

$$\begin{aligned}\mathcal{P}\{p\}(k_x,k_y,k_z,\omega) &= \\ \int_{t_0=t_i}^{t^+}\oiint_{\mathbf{r}_0\in\mathcal{S}} \Big[\nabla_0 p(\mathbf{r}_0,t_0) &\iiiint_{(\mathbf{r},t)\in\mathbb{R}^4} G(\mathbf{r},t|\mathbf{r}_0,t_0)\, e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)}\mathrm{d}^3\mathbf{r}\;\mathrm{d}t \\ -p(\mathbf{r}_0,t_0) &\iiiint_{(\mathbf{r},t)\in\mathbb{R}^4} \nabla_0 G(\mathbf{r},t|\mathbf{r}_0,t_0)\, e^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)}\mathrm{d}^3\mathbf{r}\;\mathrm{d}t\Big]\cdot\mathrm{d}\mathbf{S}_0\;\mathrm{d}t_0\end{aligned} \tag{48}$$

For the second term of the integrand, the gradient ∇_0 and the quadruple integral can be inverted, since they do not apply to the same set of variables, so that the above equation comes down to:

$$\begin{aligned}\mathcal{P}\{p\}(k_x,\, k_y,\, k_z,\, \omega) &= \\ \int_{t_0=t_i}^{t^+}\oiint_{\mathbf{r}_0\in\mathcal{S}} \Big[\nabla_0 p(\mathbf{r}_0,t_0) &< G_{\mathbf{r}_0,t_0}|\Psi^{\mathrm{pw}}_{k_x,k_y,k_z,\omega}> \\ -p(\mathbf{r}_0,t_0)\,\nabla_0 &< G_{\mathbf{r}_0,t_0}|\Psi^{\mathrm{pw}}_{k_x,\, k_y,\, k_z,\, \omega}>\Big]\cdot\mathrm{d}\mathbf{S}_0\;\mathrm{d}t_0\end{aligned} \tag{49}$$

Similar expressions are obtained for cylindrical harmonics and spherical harmonics, replacing $\Psi^{\mathrm{pw}}_{k_x,\, k_y,\, k_z,\, \omega}$ in the above equation by $\Psi^{\mathrm{ch}}_{k_r,\, l,\, k_z,\, \omega}$ and $\Psi^{\mathrm{sh}}_{k,\, l,\, m,\, \omega}$ respectively.

Thus, the key point of this type of conversion lies in the calculus of the inner products. These are directly calculated from the wave equation (17) satisfied by Green's function. We apply the analysis operator of the harmonic representation considered. Several simplifications occur by doing this operation, because plane waves, cylindrical and spherical harmonics are eigenvectors of the Laplacian operator with the eigenvalues $-\left(k_x^2+k_y^2+k_z^2\right)$ for plane waves, $-\left(k_r^2+k_z^2\right)$ for cylindrical harmonics, and $-k^2$ for spherical harmonics [12]. Moreover, they are also eigenvectors of the second time derivative operator, with eigenvalue $-\omega^2$. The result of analysis operators applied to the right hand side of the equation $-\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(t-t_0)$ is obvious from the definition of Dirac's delta function. We obtain the final results for the inner products [12]:

$$< \mathrm{G}_{\mathbf{r}_0,t_0} | \Psi_{k_x,\, k_y,\, k_z,\, \omega} > \;=\; \frac{e^{-\mathrm{i}\left(k_x x_0 + k_y y_0 + k_z z_0 + \omega t_0\right)}}{k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{c^2}} \tag{50}$$

$$< \mathrm{G}_{\mathbf{r}_0,t_0} | \Psi_{k_r,\, l,\, k_z,\, \omega} > \;=\; \frac{\mathrm{J}_l\left(k_r r_0\right) e^{-\mathrm{i}\left(l\phi_0 + k_z z_0 + \omega t_0\right)}}{k_r^2 + k_z^2 - \frac{\omega^2}{c^2}} \tag{51}$$

$$< \mathrm{G}_{\mathbf{r}_0,t_0} | \Psi_{k,\, l,\, m,\, \omega} > \;=\; \frac{\mathrm{j}_l\left(k r_0\right) \overline{\mathrm{Y}_l^m}\left(\phi_0, \theta_0\right) e^{-\mathrm{i}\omega t_0}}{k^2 - \frac{\omega^2}{c^2}} \tag{52}$$

These results are examples of the application of generalized Fourier transforms to deal with elementary inhomogeneous problems, which have been pointed out in section 3.4. Contrary to the homogeneous problems where the generalized Fourier transforms were non-null only when the dispersion relationships (5), (10) and (15) were satisfied, the generalized Fourier transforms of Green's functions are non-null even if these dispersion relationships are not satisfied. Nevertheless, it is noticed that the information is given by the location of the singularities of the generalized Fourier transforms, which occur only when the dispersion relationships are verified.

5.2.2 Projection of harmonic sound field representations on integral ones

If either one of the harmonic representations of a sound field $\mathcal{P}\{p\}(k_x,k_y,k_z,\omega)$, $\mathcal{C}\{p\}(k_r,l,k_z,\omega)$ or $\mathcal{S}\{p\}(k,l,m,\omega)$ is known, the synthesis operators (23), (28) and (33) enable us to compute the time-space profile $p(\mathbf{r},t)$ of the sound field. From this profile, it is possible to deduce the parameters required for the integral representation, given by equation (37), since the field is known everywhere. This strategy enables us to perform the "conversion" between these two kinds of representations, but there are several limitations:

- Kirchhoff's integral is only able to synthesize sound fields that satisfy the wave equation (1), whereas harmonic descriptions are able to represent more general fields.
- For a particular domain $\mathcal{V}$, Kirchhoff's form of Huygens' principle is unable to synthesize sources that are inside this domain, contrary to the harmonic representations, which are valid in $\mathbb{R}^3$.

For these reasons, this "conversion" is destructive, and it is more appropriate to call it a projection.

First limitation

Harmonic descriptions encompass a wider scope than Kirchhoff's integral-based representation, so that one can wonder how Kirchhoff's integral (36) behaves when the parameters provided by equation (37) from a harmonic description do not correspond to a field which satisfies the wave equation (1). This question is pertinent because it raises the problem of the relevance of the model satisfied by sound fields. Equation (1) is an approximation of the reality. For instance, this equation does not take into account the absorption in the propagation medium. Moreover, the consequences of an error of sound velocity on this model, which can be pointed out using a plane wave with wave number $k = \omega/(c+\delta c)$ as a reference field, providing the set of parameters (37), and then using (36) to resynthesize the sound field, deserve to be studied.

Admittedly, it can be argued that this is a well refined context, and that the subjacent errors can be neglected. Nevertheless, this kind of limitation will naturally occur when dealing with discrete observations of the sound field, by means of microphones. Indeed, the sampled sound field does not satisfy at all equation (1), unless sampling theorems are used in order to reconstruct the analog version of the sound field. It would require too high a density of microphones in practice. The effects of discretization of Kirchhoff's integral have already been studied by Berkhout and al.[15], leading to the phenomenon of *spatial aliasing*.

Second limitation

In this paragraph, we will deepen our knowledge of how Kirchhoff's form of Huygens' principle (36) works. For this purpose, we will not use the assumptions made for its derivation, that is null initial conditions and no sources inside the domain $\mathcal{V}$. Instead, we choose a closed surface $\mathcal{S}$ delimiting an exterior domain $\mathcal{V}_{\text{ext}}$ and an interior domain $\mathcal{V}_{\text{int}}$, and we suppose that there are no sources present on this surface. This surface is considered as a continuous sensor, recording the values of the pressure and its gradient at any time. Then this surface sensor is replaced by a continuous distribution of dipoles and monopoles fed by the corresponding pressure and its normal derivative respectively. The objective of this paragraph is to answer the question of which sound field is effectively synthesized. Two cases will be studied to achieve this goal: the case of exterior sources will be first investigated, and then the case of interior sources. Finally, the synthesis of these two cases is made to answer the initial question. In all this development, the surface vector $\mathrm{d}\mathbf{S}_0$ is supposed to point outward, and the sound field and its gradient are supposed to vanish at infinity.

In the case where only exterior sources are present, Kirchhoff's form of Huygens' principle can be applied for the interior domain, where the hypotheses are satisfied. This integral equation says that the sound field is correctly synthesized in $\mathcal{V}_{\text{int}}$, while it is null in $\mathcal{V}_{\text{ext}}$. In the following, the component of the acoustic pressure field created by exterior sources is denoted by $p_{\text{ext}}(\mathbf{r},t)$.

In the case where only interior sources are present, the same conclusion should apply, except that we have chosen to use a surface vector $\mathrm{d}\mathbf{S}_0$ which points outward, whereas the result would have been valid only if this vector pointed inward. So, the consequence is that the sound field synthesized is $-p(\mathbf{r},t)$ in $\mathcal{V}_{\text{ext}}$ and is null in $\mathcal{V}_{\text{int}}$. In the following , the component of the acoustic pressure field created by interior sources is denoted by $p_{\text{int}}(\mathbf{r},t)$.

In the general case where sources are both present in the interior and exterior domains, but not on the boundary, the sound field synthesized by Kirchhoff's integral equation, with $\mathrm{d}\mathbf{S}_0$ pointing outward, is $p_{\text{ext}}(\mathbf{r},t)$ in the interior domain $\mathcal{V}_{\text{int}}$ and $-p_{\text{int}}(\mathbf{r},t)$ in the exterior domain $\mathcal{V}_{\text{ext}}$. This statement is the main limitation of the projection from a harmonic representation of the sound field to an integral one. Indeed, if the sound field is synthesized using either (23), (28) or (33) to provide the parameters required to compute Kirchhoff's integral (37), the sound field effectively synthesized is not the initial one $p(\mathbf{r},t)$, but is as described in the previous paragraph. The initial harmonic representation does not distinguish the interior and exterior component of the sound field, contrary to Kirchhoff's form of Huygens' principle.

6 Plane Waves analysis inside a sphere

All current methods of sound field analysis have been reviewed in the previous parts of this article. In this section, we propose to give different descriptions of the same sound field event: the synthesis of plane waves $\Psi^{\mathrm{pw}}_{k_x,k_y,k_z,\omega}$ satisfying the dispersion relationship (5) inside a sphere of radius R, denoted by $\mathcal{V}$. So, the space-time profile of the reference field is given by the relation:

$$p_{\mathrm{ref}}(\mathbf{r},t) = e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)}.W(r) \quad \text{with} \begin{cases} W(r)=1 & \text{if } r \leq R \\ W(r)=0 & \text{if } r > R \end{cases} \tag{53}$$

In this part, the vectors $\mathbf{k}$ and $\mathbf{r}$ will be indifferently expressed either in cartesian coordinates (k_x,k_y,k_z) and (x,y,z) or in spherical coordinates (k,ϕ_k,θ_k) and (r,ϕ_r,θ_r).

This reference sound field will be analyzed using the spherical harmonic transform (30) in paragraph 6.1, using Kirchhoff's integral (37) in paragraph 6.2, and using the multidimensional Fourier transform (21) in paragraph 6.3. The cylindrical harmonic transform is not considered because it is less appropriate to the spherical geometry of the domain. Then, in the last paragraph, the assets and drawbacks of each representation are summarized.

6.1 Spherical harmonics description

6.1.1 Spherical harmonic transform of the reference sound field

In this paragraph, the spherical harmonic transform (30) of the reference sound field $p_{\mathrm{ref}}(\mathbf{r},t)$ is derived. Substituting (53) into (30) gives:

$$\begin{aligned} &\mathcal{S}\{p_{\mathrm{ref}}\}(K,L,M,\Omega) = \\ &\quad \iiiint_{(\mathbf{r},t)\in\mathbb{R}^4} W(r)\, e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)}\overline{\Psi^{\mathrm{sh}}_{K,L,M,\Omega}}(\mathbf{r},t)\ \mathrm{d}^3\mathbf{r}\ \mathrm{d}t \end{aligned} \tag{54}$$

This integral will be expressed in spherical coordinates $\mathrm{d}^3\mathbf{r} = r^2\ \mathrm{d}r\ \mathrm{d}\phi_r \sin\theta_r\ \mathrm{d}\theta_r$. For this purpose, the plane wave $e^{\mathbf{ik}\cdot\mathbf{r}}$ is expanded into classical spherical harmonics using identity (70). This yields:

$$\begin{aligned} &\mathcal{S}\{p_{\mathrm{ref}}\}(K,L,M,\Omega) = \\ &4\pi\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} \mathrm{i}^l \int_{r=0}^{R} \mathrm{j}_l(kr)\,\mathrm{j}_l(Kr)\, r^2\ \mathrm{d}r \int_{t=-\infty}^{+\infty} e^{\mathrm{i}(\omega-\Omega)t}\ \mathrm{d}t \\ &\quad .\int_{\phi_r=0}^{2\pi}\int_{\theta_r=0}^{\pi} \overline{\mathrm{Y}_l^m}(\phi_k,\theta_k)\,\mathrm{Y}_l^m(\phi_r,\theta_r)\,\overline{\mathrm{Y}_L^M}(\phi_r,\theta_r)\,\mathrm{d}\phi_r \sin\theta_r \mathrm{d}\theta_r \end{aligned} \tag{55}$$

The radial integral stands up to $r = R$ because of the window $W(r)$. The result of this integral is given by formula (71) of the appendix. Using also the orthogonality of classical spherical harmonics (67) and the orthogonality of complex exponentials (63), the above equation comes down to:

$$\begin{aligned} &\mathcal{S}\{p_{\mathrm{ref}}\}(K,L,M,\Omega) = 8\pi^2 \mathrm{i}^L \overline{\mathrm{Y}_L^M}(\phi_k,\theta_k)\,\delta(\Omega-\omega) \\ &\quad .\begin{cases} \dfrac{R^2}{k^2-K^2}[K\,\mathrm{j}_{L-1}(KR)\,\mathrm{j}_L(kR) - k\,\mathrm{j}_{L-1}(kR)\,\mathrm{j}_L(KR)] & \text{if } k \neq K \\ \dfrac{R^3}{2}\left[[\mathrm{j}_L(kR)]^2 - \mathrm{j}_{L-1}(kR)\,\mathrm{j}_{L+1}(kR)\right] & \text{if } k = K \end{cases} \end{aligned} \tag{56}$$

This equation is to be compared to the plane waves expansion into spherical harmonics (44). Indeed, the only change is that the plane wave is observed on a finite horizon, until $r = R$. On an infinite horizon, the integral involving the radial variable r would have lead to the closure relation of spherical Bessel functions (66), giving the final result (44).

In the same manner as the spectrum is widened in the Fourier domain when a monodimensional signal is observed on a finite horizon —the windowing effect— the spectrum is widened here for the variable K, because of a finite observation relative to the variable r. Thus, it is interesting to plot the spherical Hankel transform of order L of the function $W(r)\,\mathrm{j}_L(kr)$, which corresponds to the result of Lommel's integral on the right of the brace in the previous equation (56).

In figure 2, the magnitude of the spherical Hankel transform of $W(r)\,\mathrm{j}_L(kr)$ (remember, k is fixed, K is variable) is displayed for the order $L = 0$, and for several radii R, from 1m to 100m. The spectrum has been multiplied by K in order to conform to the Parseval relation for spherical Hankel transform, so that the figure looks like a classical Fourier spectrum. It is verified that the bigger R is, the better the resolution is because the spread of the main lobe decreases. As R increases, the spherical Hankel spectrum tends to $\delta(k-K)$.

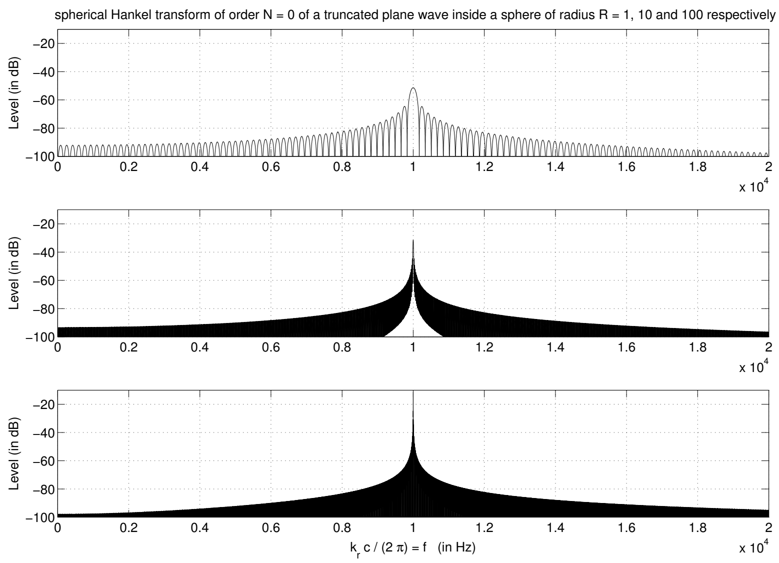

Figure 2: Spherical Hankel transform of order $L = 0$ of the function $W(r)\,\mathrm{j}_L(Kr)$ for the radii $R = 1$ (top), 10 (middle), and 100 (bottom). The plane wave has a value of $k = 2\pi\ 10000/c$. The $K-$axis is graduated in frequency from the dispersion relationship $K = 2\pi f/c$.

The second interesting effect to be observed is to fix R to a given value ($R = 1$m here), and to study the effects of the variation of the order L of the spherical Hankel transform of $W(r)\,\mathrm{j}_L(kr)$. The result is displayed in figure 3. It is shown that the global level of power decays when the order L increases, which conforms to the theory. Indeed, it is often seen in the literature that spherical harmonics are excited until an order $L \sim kR$ (1.84 with $k = 2\pi f/c$, $f = 100$Hz and $R = 1$m).

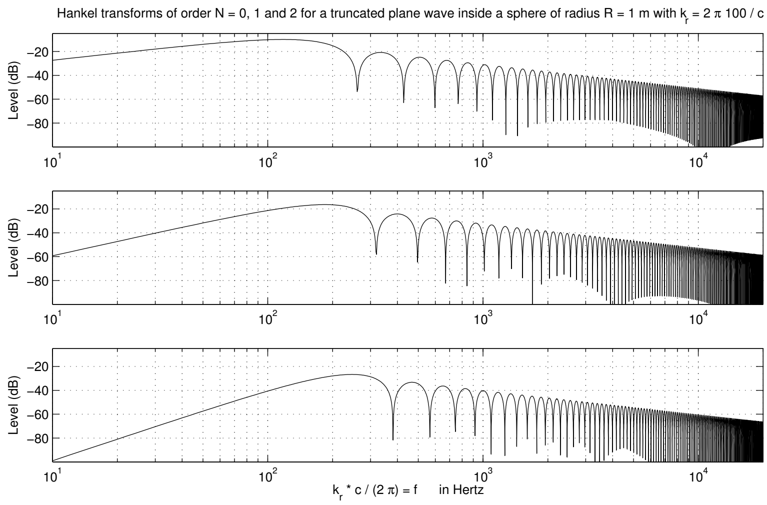

Figure 3: Spherical Hankel transform of order $L = 0$ (top), $L = 1$ (middle), and $L = 2$ (bottom) of $W(r)\mathrm{j}_L(kr)$ inside a sphere of radius $R = 1\mathrm{m}$. The plane wave has a value of $k = 2\pi\ 100/c$ and the $K-$axis is graduated in frequency from the dispersion relationship $K = 2\pi f/c$.

6.1.2 Link to modal analysis

In the field of modal analysis [1] and modal control [16], the sound field is generally expanded on a finite discrete set of modes. This is made possible because of the finiteness of the region being analyzed, which is the case for our reference sound field.

Indeed, for instance, a time signal observed on a finite horizon can be decomposed into Fourier series. The discrete set of complex exponentials used for the expansion is orthogonal. Moreover, if the signal is supposed to be bandlimited, then, only a finite number of modes is required to model the signal. Nevertheless, note the paradox: a time signal cannot be finite in the time domain and in the frequency domain. However, it is supposed that the truncation of the Fourier series to a finite number of modes does not induce a large square error.

In our case, if we only consider the spatial dependency of the sound field, this one is analyzed as the product of a spherical Hankel transform and a decomposition into classical spherical harmonics, as stated by equation (30). The decomposition into classical spherical harmonics is naturally discrete. And, in the same manner as a finite time signal can be decomposed into Fourier series, the finite radius R of the sphere enables us to decompose the radial dependency of the signal into generalized Fourier series: for our case, they are spherical Fourier-Bessel series. The elementary atoms are not complex exponentials anymore, but $\mathrm{j}_l(\alpha_{ln}r)$, where α_{ln} is the n^{th} root of the equation $\mathrm{J}_{l+1/2}(\alpha r) = 0$ (see Poularikas [17] for Fourier-Bessel series). So, inside this sphere, the sound field can be approximated by the following series:

$$p(\mathbf{r}) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} \sum_{n=1}^{+\infty} \alpha_{lmn} \mathrm{j}_l(\alpha_{ln}) \mathrm{Y}_l^m(\phi, \theta) \tag{57}$$

This set of solutions is orthogonal for the spherical inner product:

$$< f|g >= \int_{r=0}^{R} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} f(\mathbf{r})\, \overline{g}(\mathbf{r})\, r^2 \,\mathrm{d}r \,\mathrm{d}\phi \sin\theta \,\mathrm{d}\theta \tag{58}$$

This expansion can be truncated to a finite number of terms, if a tolerance is imposed on the error between the initial signal and the sum of weighted modes. For this purpose, figure 2 and 3 are useful, because they enable us to calculate the weights associated to the spherical Fourier-Bessel series for a given wave number k.

6.2 Kirchhoff's integral description

To reproduce the reference sound field inside the domain $\mathcal{V}$, the acoustic pressure and its gradient are required on the sphere of radius R in order to be able to compute Kirchhoff's form of Huygens' principle (36). These are given by the relations:

$$\begin{aligned} p(R,\phi_0,\theta_0,t_0) &= e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}_0+\omega t_0)} \\ \nabla p(R,\phi_0,\theta_0,t_0) &= \mathrm{i}\, e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}_0+\omega t_0)}\, \mathbf{k} \end{aligned}$$

The integration is made on the sphere, so that only ϕ_0 and θ_0 are variable, while r remains fixed to R.

Concerning the sound field effectively synthesized with these parameters, it can be said from section 5.2 that the plane wave is correctly synthesized inside the sphere. The plane wave $\Psi^{\mathrm{pw}}_{k_x,k_y,k_z,\omega}$ does not fulfill the assumption that it vanishes at infinity, so that it could not be reproduced in the exterior domain.

6.3 Plane wave description

In this paragraph, the multidimensional Fourier transform (21) of the reference sound field $p_{\mathrm{ref}}(\mathbf{r},t)$ is derived. Substituting (53) into (21) gives:

$$\begin{aligned} &\mathcal{F}\{p_{\mathrm{ref}}\}(K_x,K_y,K_z,\Omega) = \\ &\quad \iiiint_{(\mathbf{r},t)\in\mathbf{R}^4} \left[W(r)\,.e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)}\right] e^{-\mathrm{i}(\mathbf{K}\cdot\mathbf{r}+\Omega t)} \mathrm{d}^3 r \,\mathrm{d}t \end{aligned} \tag{59}$$

The previous equation is the Fourier transform of the product of $W(r)$ by $e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}+\omega t)}$. Using the property that the Fourier transform transforms a simple product into a convolution product, the last equation becomes:

$$\begin{aligned} &\mathcal{F}\{p_{\mathrm{ref}}\}(K_x,K_y,K_z,\Omega) = \\ &\quad \left(\mathcal{F}\{W\} * \mathcal{F}\left\{\Psi^{\mathrm{pw}}_{k_x,k_y,k_z,\omega}\right\}\right)(K_x,K_y,K_z,\Omega) \end{aligned} \tag{60}$$

where $*$ denotes the convolution product. The multidimensional Fourier transform of the plane wave $\Psi^{\mathrm{pw}}_{k_x,k_y,k_z,\omega}$ is $(2\pi)^4\,\delta(\mathbf{K}-\mathbf{k})\,\delta(\Omega-\omega)$. Now, we focus on the Fourier transform of W. Since W is only dependent of r, it is again useful to expand the plane wave inside the Fourier transform integral into spherical harmonics with the identity (70). This yields:

$$
\begin{aligned}
\mathcal{P}\{W\}(K_x,K_y,K_z,\Omega) = 4\pi \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} \mathrm{i}^l \\
\int_{r=0}^{+\infty} W(r)\, \mathrm{j}_l(Kr)\, r^2 \,\mathrm{d}r \int_{t=-\infty}^{+\infty} e^{-\mathrm{i}\Omega t} \mathrm{d}t \\
\int_{\phi_r=0}^{2\pi} \int_{\theta_r=0}^{\pi} \overline{\mathrm{Y}_l^m}(\phi_K,\theta_K)\, \mathrm{Y}_l^m(\phi_r,\theta_r)\, \mathrm{d}\phi_r \sin\theta_r \,\mathrm{d}\theta_r
\end{aligned}
\tag{61}
$$

The integral involving the classical spherical harmonics is non null only for the constant term, obtained for $l = m = 0$. The radial window $W(r)$ is uniform, truncating the integral on the radial variable up to $r = R$. Finally, equation (61) comes down to:

$$
\begin{aligned}
\mathcal{S}\{W\}(K_x,K_y,K_z,\Omega) &= 4\pi \int_{r=0}^{R} \mathrm{j}_0(Kr)\, r^2 \,\mathrm{d}r \;.\; 2\pi\delta(\Omega-\omega) \\
&= 4\pi \int_{r=0}^{R} \frac{\sin(Kr)}{Kr} r^2 \,\mathrm{d}r \;.\; 2\pi\delta(\Omega-\omega) \\
&= \frac{4}{3}\pi R^3 .3 \left[\frac{\sin(KR)}{(KR)^3} - \frac{\cos(KR)}{(KR)^2} \right] \;.\; 2\pi\delta(\Omega-\omega)
\end{aligned}
\tag{62}
$$

where $K = \sqrt{K_x^2 + K_y^2 + K_z^2}$. Again, as in paragraph 6.1, if the observation horizon had been infinite, then the Fourier spectrum would have been that of the plane wave, that is Dirac's delta function. The finite observation horizon introduces a windowing effect, that is finite resolution characterized by the spread of the main lobe, and also the presence of side lobes. The dependence in K of the magnitude of the Fourier transform of the spatial window $W(r)$ is plotted in figure 4. This spectrum is to be compared to a basic uniform cubic window, which would have lead to a "sinus cardinal" Fourier transform. Here the sidelobe is approximately 21dB under the main lobe, compared to 13dB for the "sinus cardinal" window. The side lobes are more attenuated because a sphere is smoother than a cube.

Moreover, the distortion introduced by the finiteness of the observation is the same for all the plane waves being analyzed, because the initial Fourier transform is distorted by the convolution product with the analysis window. On the contrary, the simple product is not transformed into a convolution product by the spherical Hankel transform (see paragraph 6.1), so that the associated distortion is dependent on the wave number k of the input plane wave to be analyzed.

6.4 Assets and drawbacks of these representations

In the three previous paragraphs, three different descriptions of the same sound field event have been presented. The spherical harmonic description of section 6.2 is used in ambisonics systems. the advantages of ambisonics systems are their ease of coding and decoding[3][18], whereas their disadvantage is that there is no direct spherical harmonic generator. In early ambisonics systems, loudspeakers were assumed to be plane wave synthesizers, which is true if the far-field hypothesis is satisfied. In more recent ambisonics systems, the coding of point sources has been improved in order to take into account the distance of the sources[19], enabling a better sound field rendering scheme. The Kirchhoff-based description of section 6.2 is used in Wave Field Synthesis systems. The assets and drawbacks of Wave Field Synthesis systems are

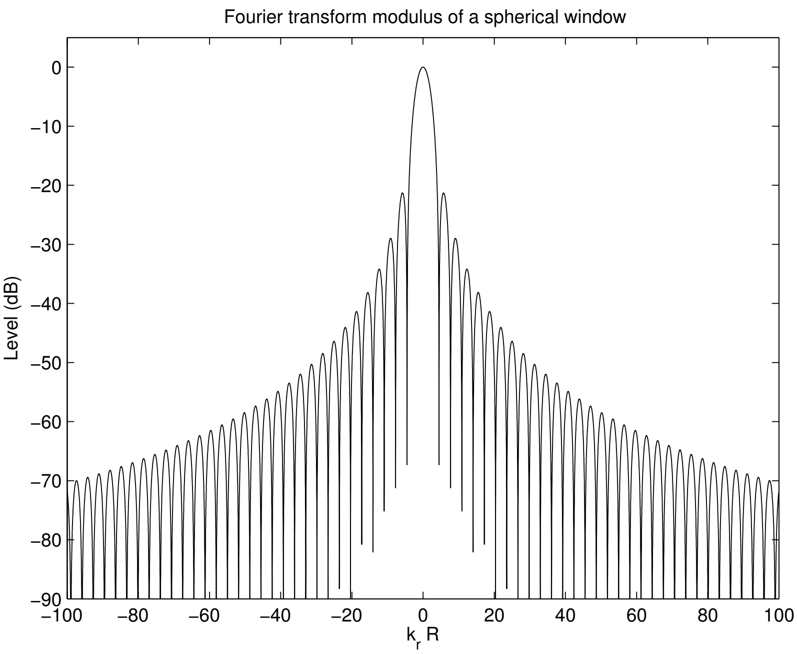

Figure 4: Fourier transform of a uniform spherical window normalized by $\frac{4}{3}\pi R^3$, that is $3\left[\frac{\sin(KR)}{(KR)^3} - \frac{\cos(KR)}{(KR)^2}\right]$

inverted compared to ambisonics systems. Indeed, Wave Field Synthesis systems use monopoles and dipoles for the synthesis which could be approximated by real sources in practice, providing an easy rendering scheme. On the other hand, their disadvantage is the complexity of coding and decoding, compared to ambisonics systems.

The last model described in section 6.3 has not its own dedicated sound reproduction method yet, and appears to be an interesting perspective. It has much in common with ambisonics systems, except for the another harmonic description, so that they mainly share the same assets and drawbacks. Nevertheless, using planes waves as harmonics instead of spherical harmonics has the additional asset pointed out in the previous paragraph, that is the distortion due to a finite observation horizon is independent of the harmonic being analyzed, contrary to spherical harmonics, for which the distortion is dependent on the wavenumber k.

7 Conclusion

Several efficient representations of a sound field have been presented in this article. Harmonic representations decompose the initial sound field into a set of harmonics, either plane waves, cylindrical harmonics, or spherical harmonics. These harmonics form an orthogonal set, which moreover spans the whole set of sound fields, and more. The mathematical tools for these decompositions are generalized Fourier transforms. The representation based on Kirchhoff's integral equation has also been presented, and particularly Kirchhoff's form of Huygens' principle. This integral equation synthesizes the sound field radiated by exterior sources in the interior domain and reciprocally. The integral representation is only able to synthesize sound fields which rigorously satisfy the wave equation, whereas slight errors in the sound field model do not affect much harmonic descriptions. The equivalence between all harmonic descriptions has been demonstrated. Moreover, some links between harmonic descriptions and the integral representation have been established: given that the set spanned by the harmonics is larger than the one spanned by Kirchhoff's integral, the transition from the integral representation to a harmonic description is always feasible, whereas the inverse operation is a projection, so that equivalence is only assured under some restrictions which have been pointed out.

A variety of viewpoints have been explored for the case truncated plane waves inside a sphere of radius R. The spherical harmonic description is then linked to the traditional High Order Ambisonics approach; Kirchhoff's integral equation model is linked to the traditional Wave Field Synthesis approach; on the contrary, the plane wave description is not linked to any sound field reproduction system yet and thus appears as an interesting perspective of this work. Although the equivalence of these different representations has been shown in this study under ideal conditions, this does not hold when this assumption is not met, which occurs when dealing with real recordings made by an array of microphones for instance. Some further works are necessary to study the behavior of these representations when facing these non ideal conditions.

A Mathematical results

This appendix lists some useful mathematical identities involving the Dirac delta function.

The Fourier transform of a complex exponential provides the basic identity:

$$\int_{x=-\infty}^{+\infty} e^{\mathrm{i}(k_1-k_2)x}\mathrm{d}x = 2\pi\delta(k_1-k_2) \tag{63}$$

Bessel functions of the first kind satisfy the *closure relation*:

$$\int_{r=0}^{+\infty} \mathrm{J}_l(\alpha r)\,\mathrm{J}_l(\beta r)\, r\, \mathrm{d}r = \frac{\delta(\alpha-\beta)}{\sqrt{\alpha\beta}} = \frac{\delta(\alpha-\beta)}{\beta} = \frac{\delta(\alpha-\beta)}{\alpha} \tag{64}$$

The spherical Bessel functions of the first kind $\mathrm{j}_l(x)$ are linked to the Bessel functions of the first kind $\mathrm{J}_l(x)$ with the formula:

$$\mathrm{j}_l(x) = \sqrt{\frac{2}{\pi x}}\mathrm{J}_{l+1/2}(x) \tag{65}$$

From this last equation, we deduce the *closure relation* for the spherical Bessel functions of the first kind:

$$\int_{r=0}^{+\infty} \mathrm{j}_l(\alpha r)\,\mathrm{j}_l(\beta r)\, r^2\, \mathrm{d}r = \frac{2}{\pi}\frac{\delta(\alpha-\beta)}{\alpha\beta} = \frac{2}{\pi}\frac{\delta(\alpha-\beta)}{\alpha^2} \tag{66}$$

The spherical harmonics $\mathrm{Y}_l^m(\theta,\phi)$ are orthogonal:

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \mathrm{Y}_{l_1}^{m_1}(\phi,\theta)\,\overline{\mathrm{Y}_{l_2}^{m_2}}(\phi,\theta)\,\mathrm{d}\phi\ \sin\theta\ \mathrm{d}\theta = \delta_{l_1 l_2}\delta_{m_1 m_2} \tag{67}$$

The first derivative of the Dirac delta function satisfies this property:

$$\int_{x=-\infty}^{+\infty} p(x)\,\delta'(x-x_0)\,\mathrm{d}x = -p'(x_0) \tag{68}$$

The identity of Jacobi-Anger is:

$$e^{\mathrm{i}kr\cos\phi} = \sum_{l=-\infty}^{+\infty} \mathrm{i}^l \mathrm{J}_l(kr)\, e^{\mathrm{i}l\phi} \tag{69}$$

An analogous identity to the Jacobi-Anger one, involving the spherical Bessel functions characterizes the expansion of a plane wave in classical spherical harmonics, using $\mathbf{r} = (r,\phi_r,\theta_r)$ and $\mathbf{k} = (k,\phi_k,\theta_k)$ in spherical coordinates:

$$e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = 4\pi\sum_{l=0}^{+\infty}\sum_{m=-l}^{l} \mathrm{i}^l \mathrm{j}_l(kr)\,\overline{\mathrm{Y}_l^m}(\phi_k,\theta_k)\,\mathrm{Y}_l^m(\phi_r,\theta_r) \tag{70}$$

Lommel's definite integral involving products of spherical Bessel of the same order is given by the following formula:

$$\int \mathrm{j}_l(\alpha r)\,\mathrm{j}_l(\beta r)\, r^2\, \mathrm{d}r = \\ \begin{cases} \dfrac{r^2}{\alpha^2-\beta^2}\left[\beta \mathrm{j}_{l-1}(\beta r)\,\mathrm{j}_l(\alpha r) - \alpha \mathrm{j}_{l-1}(\alpha r)\,\mathrm{j}_l(\beta r)\right] & \text{if } \alpha \neq \beta \\ \dfrac{r^3}{2}\left[\left[\mathrm{j}_l(\alpha r)\right]^2 - \mathrm{j}_{l-1}(\alpha r)\,\mathrm{j}_{l+1}(\alpha r)\right] & \text{if } \alpha = \beta \end{cases} \tag{71}$$

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