Performance of ESPRIT for estimating mixtures of complex exponentials modulated by polynomials : supporting document

Performances d'ESPRIT pour l'estimation de mélanges d'exponentielles complexes modulées par des polynômes : document de support

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Performance of ESPRIT for Estimating Mixtures of Complex Exponentials Modulated by Polynomials: Supporting Document

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***Abstract*—High Resolution (HR) methods are known to provide accurate frequency estimates for discrete spectra [1]. The Polynomial Amplitude Complex Exponentials (PACE) model was presented as the most general model tractable by HR methods. A subspace-based estimation scheme was recently proposed in [2], derived from the ESPRIT algorithm [3]. In [4], we focused on the performance of this estimator. We first presented some asymptotic expansions of the estimated parameters, obtained at the first order under the assumption of a high signal-to-noise ratio (SNR). Then the performance of the generalized ESPRIT algorithm for estimating the parameters of this model was analyzed in terms of bias and variance, and compared to the Cramér-Rao bounds.**

In this supporting document, we present the proofs of the theoretical results introduced in [4]. This document, written as a sequel of [4], is not intended to be read separately. It is organized as follows: section I is devoted to the perturbation analysis, then the performance of the estimators is analyzed in section II.

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***Résumé*—Les méthodes à Haute Résolution (HR) sont connues pour fournir une estimation fréquentielle précise des spectres discrets [1]. Le modèle d'Exponentielles Complexes à modulation d'Amplitude Polynomiale (PACE) a été présenté comme le modèle le plus général que l'on puisse traiter par des méthodes HR. Une procédure d'estimation de type sous-espace a été récemment proposée dans [2], développée à partir de l'algorithme ESPRIT [3]. Dans [4], nous avons porté notre attention sur les performances de cet estimateur. Nous avons d'abord présenté des développements asymptotiques des paramètres estimés, obtenus au premier ordre sous l'hypothèse d'un rapport signal-à-bruit (RSB) élevé. Les performances de l'algorithme ESPRIT généralisé visant à estimer les paramètres de ce modèle ont ensuite été analysées en terme de biais et de variance, et comparées aux bornes de Cramér-Rao. Dans ce document de support, nous présentons les preuves des résultats théoriques introduits dans [4]. Ce document, conçu pour être lu en complément de [4], est structuré de la façon suivante : la section I porte sur l'analyse des perturbations, puis les performances des estimateurs sont analysées dans la section II.**

***Index Terms*—ESPRIT, high resolution (HR), multiple eigenvalues, performance analysis, perturbation theory, polynomial modulation.**

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I. Perturbation analysis

A. Perturbation of the spectral matrix

Proof of proposition 10: First, note that the function $\varepsilon \mapsto \det\left(\boldsymbol{W}(\varepsilon)_{\downarrow}^{H}\boldsymbol{W}(\varepsilon)_{\downarrow}\right)$ is continuous. As $\boldsymbol{W}_{\downarrow}$ is full-rank, it is non-zero at $\varepsilon = 0$. Thus it remains non-zero in a neighborhood of 0. Consequently, $\boldsymbol{W}_{\downarrow}(\varepsilon)$ is full-rank in this neighborhood. Moreover, according to proposition 9, the function $\varepsilon \mapsto \boldsymbol{W}(\varepsilon)$ is $\mathcal{C}^{\infty}$. Thus the function $\boldsymbol{\Phi}(\varepsilon) = \left(\boldsymbol{W}(\varepsilon)_{\downarrow}^{H}\boldsymbol{W}(\varepsilon)_{\downarrow}\right)^{-1}\boldsymbol{W}(\varepsilon)_{\downarrow}^{H}\boldsymbol{W}(\varepsilon)_{\uparrow}$ is also $\mathcal{C}^{\infty}$. Thus equations (34) to (36) can be derived by calculating the first order expansion of the equality

$$\left(\boldsymbol{W}(\varepsilon)_{\downarrow}^{H}\boldsymbol{W}(\varepsilon)_{\downarrow}\right)\boldsymbol{\Phi}(\varepsilon) = \boldsymbol{W}(\varepsilon)_{\downarrow}^{H}\boldsymbol{W}(\varepsilon)_{\uparrow}.$$

■

B. Perturbation of the Jordan matrix

Proof of corollary 11: Left-multiplying equation (34) by $\boldsymbol{G}^{-1}$ and right-multiplying it by $\boldsymbol{G}$ yields (37) and (38), where

$$\Delta\boldsymbol{J}^{\perp} = \boldsymbol{G}^{-1}\Delta\boldsymbol{\Phi}^{\perp}\boldsymbol{G}. \tag{55}$$

Left-multiplying equation (36) by $\boldsymbol{G}^{-1}$, right-multiplying it by $\boldsymbol{G}$, and substituting equations (55), (7), and (8) yields

$$\Delta\boldsymbol{J}^{\perp} = -\boldsymbol{V}_{\downarrow}^{n\dagger}\Delta\boldsymbol{W}_{\downarrow}^{\perp}\boldsymbol{G}\,\boldsymbol{J} + \boldsymbol{J}\,\boldsymbol{V}_{\uparrow}^{n\dagger}\Delta\boldsymbol{W}_{\uparrow}^{\perp}\boldsymbol{G}. \tag{56}$$

In other respects, equation (4) yields $\boldsymbol{S}^{\dagger} = \boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}\boldsymbol{V}^{n\dagger}$. Thus substituting equations (29) and (7) into (56) yields

$$\begin{aligned}\Delta\boldsymbol{J}^{\perp} \;=\;& -\boldsymbol{V}_{\downarrow}^{n\dagger}\left(\boldsymbol{I}_{n\downarrow} - \boldsymbol{V}_{\downarrow}^{n}\boldsymbol{V}^{n\dagger}\right)\Delta\boldsymbol{S}\,\boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}\boldsymbol{J}\\ &+\boldsymbol{J}\,\boldsymbol{V}_{\downarrow}^{n\dagger}\left(\boldsymbol{I}_{n\uparrow} - \boldsymbol{V}_{\uparrow}^{n}\boldsymbol{V}^{n\dagger}\right)\Delta\boldsymbol{S}\,\boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}.\end{aligned}$$

From this last equation, lemma 15 below proves equation (39). ■

Lemma 15. *The Pascal-Vandermonde matrix $\boldsymbol{V}^{n}$ satisfies the following identities:*

$$\begin{bmatrix}\boldsymbol{V}_{\downarrow}^{n\dagger}, & \boldsymbol{0}_{(n\times 1)}\end{bmatrix} - \boldsymbol{V}^{n\dagger} = -\boldsymbol{v}'_{\downarrow}\,{\boldsymbol{e}'_{\downarrow}}^{H} \tag{57}$$

$$\begin{bmatrix}\boldsymbol{0}_{(n\times 1)}, & \boldsymbol{V}_{\uparrow}^{n\dagger}\end{bmatrix} - \boldsymbol{V}^{n\dagger} = -\boldsymbol{v}'_{\uparrow}\,{\boldsymbol{e}'_{\uparrow}}^{H}. \tag{58}$$

where

- $\boldsymbol{v}'_{\downarrow} = \frac{\boldsymbol{Z}^{-1}\boldsymbol{v}_{\downarrow}}{1-\boldsymbol{v}_{\downarrow}^{H}\boldsymbol{Z}^{-1}\boldsymbol{v}_{\downarrow}}$
- $\boldsymbol{v}'_{\uparrow} = \frac{\boldsymbol{Z}^{-1}\boldsymbol{v}_{\uparrow}}{1-\boldsymbol{v}_{\uparrow}^{H}\boldsymbol{Z}^{-1}\boldsymbol{v}_{\uparrow}}$
- $\boldsymbol{e}_{\downarrow} = [0\ldots 0, 1]^{T} - \boldsymbol{V}^{n}\boldsymbol{Z}^{-1}\boldsymbol{v}_{\downarrow}$

- $\boldsymbol{e}_{\uparrow} = [1, 0 \dots 0]^T - \boldsymbol{V}^n \boldsymbol{Z}^{-1} \boldsymbol{v}_{\uparrow}$
- $\boldsymbol{Z} = \boldsymbol{V}^{nH} \boldsymbol{V}^n$.

Proof: Let us show equation (57). Since the matrix $\boldsymbol{V}^n_{\downarrow}$ is full-rank, $\boldsymbol{V}^{n\dagger}_{\downarrow} = \left(\boldsymbol{V}^{nH}_{\downarrow}\boldsymbol{V}^n_{\downarrow}\right)^{-1}\boldsymbol{V}^{nH}_{\downarrow}$. Besides, $\boldsymbol{V}^{nH}_{\downarrow}\boldsymbol{V}^n_{\downarrow} = \boldsymbol{Z} - \boldsymbol{v}_{\downarrow}\boldsymbol{v}_{\downarrow}^H$. Applying the matrix inversion lemma [5, pp. 18-19] to this last equality and right-multiplying it by $\boldsymbol{V}^{nH}_{\downarrow}$ yields

$$\boldsymbol{V}^{n\dagger}_{\downarrow} = \boldsymbol{Z}^{-1}\boldsymbol{V}^{nH}_{\downarrow} + \boldsymbol{v}'_{\downarrow}\boldsymbol{v}^H_{\downarrow}\boldsymbol{Z}^{-1}\boldsymbol{V}^{nH}_{\downarrow}. \quad (59)$$

Besides,

$$\boldsymbol{V}^{n\dagger} = \left[\begin{array}{cc} \boldsymbol{Z}^{-1}\boldsymbol{V}^{nH}_{\downarrow}, & \left(1 - \boldsymbol{v}^H_{\downarrow}\boldsymbol{Z}^{-1}\boldsymbol{v}_{\downarrow}\right)\boldsymbol{v}'_{\downarrow} \end{array}\right]. \quad (60)$$

Substituting equations (60) and (59) into the left member of equation (57) finally leads to the right member of equation (57). Equation (58) can be derived in a similar way. ■

C. Perturbation of the poles

Proof of proposition 12: Remember that the matrix $\boldsymbol{J}$ in equation (37) is block-diagonal, with blocks of dimensions $M_k \times M_k$. The perturbation theory shows that for all ε in the neighborhood of 0, the matrix $\boldsymbol{J}(\varepsilon)$ in equation (37) can also be block-diagonalized, with blocks of the same dimensions, and that the functions which associate each of these blocks to ε are $\mathcal{C}^\infty$. More precisely, the function which associates the $M_k \times M_k$ block related to the pole z_k to ε admits the first order expansion:

$$\boldsymbol{J}_k(\varepsilon) = \boldsymbol{J}_k + \varepsilon\left(\Delta\boldsymbol{J}^{\perp}_k + \boldsymbol{A}'_k\boldsymbol{J}_k - \boldsymbol{J}_k\boldsymbol{A}'_k\right) + O(\varepsilon^2).$$

However the sum of the eigenvalues $z_k^{(m)}(\varepsilon)$ is equal to the trace of this block. Thus the function $\varepsilon \mapsto z_k(\varepsilon)$ is $\mathcal{C}^\infty$ and admits the first order expansion

$$z_k^{(m)}(\varepsilon) = z_k + \tfrac{\varepsilon}{M_k}\,\mathrm{trace}\left(\Delta\boldsymbol{J}^{\perp}_k + \boldsymbol{A}'_k\boldsymbol{J}_k - \boldsymbol{J}_k\boldsymbol{A}'_k\right) + O(\varepsilon^2). \quad (61)$$

Since matrix products can be commuted inside the trace operator, $\mathrm{trace}\left(\boldsymbol{A}'_k\boldsymbol{J}_k - \boldsymbol{J}_k\boldsymbol{A}'_k\right) = 0$. Therefore equations (40) and (41) directly follow from equation (61). Moreover, substituting equation (39) into equation (41) shows that

$$\begin{aligned} M_k\Delta z_k &= \mathrm{trace}\left(\boldsymbol{v}'_{\downarrow k} \times \left(\boldsymbol{e}'^{\,H}_{\downarrow}\Delta\boldsymbol{S}\,\boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}\boldsymbol{J}\right)\right) \\ &- \mathrm{trace}\left(\left(\boldsymbol{J}\,\boldsymbol{v}'_{\uparrow k}\right) \times \left(\boldsymbol{e}'^{\,H}_{\uparrow}\Delta\boldsymbol{S}\,\boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}\right)\right) \\ &= \mathrm{trace}\left(\left(\boldsymbol{e}'^{\,H}_{\downarrow}\Delta\boldsymbol{S}\,\boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}\boldsymbol{J}\right) \times \boldsymbol{v}'_{\downarrow k}\right) \\ &- \mathrm{trace}\left(\left(\boldsymbol{e}'^{\,H}_{\uparrow}\Delta\boldsymbol{S}\,\boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}\right) \times \left(\boldsymbol{J}\,\boldsymbol{v}'_{\uparrow k}\right)\right) \\ &= \boldsymbol{e}'^{\,H}_{\downarrow}\Delta\boldsymbol{S}\,\boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}\boldsymbol{J}\boldsymbol{v}'_{\downarrow k} - \boldsymbol{e}'^{\,H}_{\uparrow}\Delta\boldsymbol{S}\,\boldsymbol{V}^{l\dagger T}\boldsymbol{D}^{-1}\boldsymbol{J}\boldsymbol{v}'_{\uparrow k}, \end{aligned}$$

from which equations (42) and (43) are derived[1]. ■

Proof of proposition 3: The complex logarithm is a $\mathcal{C}^\infty$-diffeomorphism of $\mathbb{C}$ into $\mathbb{R}\times]-\pi,\pi[$, thus the functions $\varepsilon \mapsto \delta_k(\varepsilon)$ and $\varepsilon \mapsto f_k(\varepsilon)$ are $\mathcal{C}^\infty$. A first order expansion yields, by using equation (40), $\ln(z_k(\varepsilon)) = \ln(z_k) + \varepsilon\dfrac{\Delta z_k}{z_k} + O\left(\varepsilon^2\right)$. Finally, equation (12) can be derived by substituting equation (44) into this first order expansion. ■

D. Perturbation of the amplitudes and phases

Proof of lemma 13: The coefficients of the matrix $\boldsymbol{V}^N(\varepsilon)$ are powers of the estimated poles $z_k(\varepsilon)$. Since these poles are $\mathcal{C}^\infty$ functions of the variable ε, the function $\varepsilon \mapsto \boldsymbol{V}^N(\varepsilon)$ is also $\mathcal{C}^\infty$. In other respects, the column of $\boldsymbol{V}^N(\varepsilon)$ related to the pole $z_k(\varepsilon)$ at index $m < M_k$ is $\frac{1}{m!}\frac{d^m\boldsymbol{v}}{dz^m}(z_k(\varepsilon))$, where $\boldsymbol{v}(z) = [1, z, \dots, z^{N-1}]^T$. Consequently, its first order expansion is

$$\begin{aligned} &\tfrac{1}{m!}\frac{d^m\boldsymbol{v}}{dz^m}(z_k) + \varepsilon\tfrac{1}{m!}\frac{d^{m+1}\boldsymbol{v}}{dz^{m+1}}(z_k)\frac{dz_k}{d\varepsilon}(0) + O(\varepsilon^2) \\ &= \overline{\boldsymbol{v}}_k^{(m)} + \varepsilon(m+1)\Delta z_k\overline{\boldsymbol{v}}_k^{(m+1)} + O(\varepsilon^2). \end{aligned}$$

where $\overline{\boldsymbol{v}}_k^{(m)}$ is the column of $\overline{\boldsymbol{V}}^N$ related to the pole z_k at index $m \leq M_k$. Thus equations (46) to (49) can be derived columnwise. ■

Proof of proposition 14: Since the poles z_k are distinct and since the functions $\varepsilon \mapsto z_k(\varepsilon)$ are continuous, they have distinct values in a neighborhood of 0. Therefore, the Pascal Vandermonde matrix $\boldsymbol{V}^N(\varepsilon)$ remains full-rank in this neighborhood. Moreover, according to lemma 13, the function $\varepsilon \mapsto \boldsymbol{V}^N(\varepsilon)$ is $\mathcal{C}^\infty$, thus the function $\varepsilon \mapsto \boldsymbol{V}^N(\varepsilon)^\dagger$ is also $\mathcal{C}^\infty$. Therefore the function $\varepsilon \mapsto \boldsymbol{\alpha}(\varepsilon) = \boldsymbol{V}^N(\varepsilon)^\dagger\boldsymbol{s}(\varepsilon)$ is $\mathcal{C}^\infty$ in a neighborhood of 0. Moreover, the first order expansion of the equality $\boldsymbol{V}^N(\varepsilon)\,\boldsymbol{\alpha}(\varepsilon) = \boldsymbol{s}(\varepsilon)$ yields

$$\boldsymbol{V}^N\Delta\boldsymbol{\alpha} + \Delta\boldsymbol{V}^N\boldsymbol{\alpha} = \Delta\boldsymbol{s}.$$

Substituting equation (47) into this last equality yields

$$\Delta\boldsymbol{\alpha} = \boldsymbol{V}^{N\dagger}\left(\Delta\boldsymbol{s} - \overline{\boldsymbol{V}}^N\Delta\boldsymbol{Z}\,\boldsymbol{\alpha}\right). \quad (62)$$

Besides, substituting equations (44), (48) and (49) into equation (62), a simple rewriting shows that

$$\Delta\boldsymbol{Z}\,\boldsymbol{\alpha} = \left[\boldsymbol{B}_0^T, \dots, \boldsymbol{B}_{K-1}^T\right]^T\Delta\boldsymbol{s} \quad (63)$$

(where the matrices $\boldsymbol{B}_k$ are defined in equation (54)). Finally, equations (52) and (53) are derived by substituting equation (63) into equation (62). ■

Proof of proposition 4: It is supposed that $|\alpha_k^{(m)}(0)| = a_k^{(m)} \neq 0$. Then since the function $\varepsilon \mapsto \alpha_k^{(m)}(\varepsilon)$ is $\mathcal{C}^\infty$, the function $\varepsilon \mapsto a_k^{(m)}(\varepsilon) = |\alpha_k^{(m)}(\varepsilon)|$ is also $\mathcal{C}^\infty$ in the neighborhood of zero. Moreover, substituting the first row of equation (13) into the equation $a_k^{(m)}(\varepsilon) = a_k^{(m)}\sqrt{\frac{\alpha_k^{(m)}(\varepsilon)}{\alpha_k^{(m)}}\,\frac{\alpha_k^{(m)}(\varepsilon)^*}{\alpha_k^{(m)*}}}$, yields its first order expansion.

In other respects, the complex logarithm is a $\mathcal{C}^\infty$-diffeomorphism of $\mathbb{C}$ into $\mathbb{R}\times]-\pi,\pi[$, thus the function $\varepsilon \mapsto \phi_k^{(m)}(\varepsilon)$ is $\mathcal{C}^\infty$. A first order expansion yields, by using equation (51),

$$\ln\left(\alpha_k^{(m)}(\varepsilon)\right) = \ln\left(\alpha_k^{(m)}\right) + \varepsilon\frac{\Delta\alpha_k^{(m)}}{\alpha_k^{(m)}} + O\left(\varepsilon^2\right)$$

from which the first order expansion of the function $\varepsilon \mapsto \phi_k^{(m)}(\varepsilon)$ can be derived. ■

[1] It can be noticed a scaling factor $\frac{1}{\alpha_k^{(M_k-1)}}$ is artificially introduced in equation (42), and counterbalanced in equation (43). This trick makes the vectors $\boldsymbol{f}_{\downarrow k}$ and $\boldsymbol{f}_{\uparrow k}$ not depend on the complex amplitudes, in the particular case of single poles.

II. Performance of the estimators

A. First order performance

The results presented in section IV-A in [4] are proved below.

Proof of proposition 5: Since the signal $\Delta s(t)$ is centered, equation (44) yields $\mathbb{E}[\Delta z_k] = 0$, thus the estimator $\widehat{z}_k$ is unbiased at the first order. Moreover, $\mathrm{var}\,(\widehat{z}_k) \sim \sigma^2\,\mathbb{E}[|\Delta z_k|^2]$ (where we have defined $\sigma = \varepsilon$), from which expression (15) can be derived by using equation (44).

In other respects, since the signal $\Delta s(t)$ is centered, substituting equation (12) into equation (11) shows that the estimators $\widehat{\delta}_k$ and $\widehat{f}_k$ are unbiased at the first order. Moreover,

$$\mathrm{var}(\widehat{\delta}_k) \sim \frac{\sigma^2}{M_k^2}\,\mathbb{E}\left[\Re\left(\frac{\boldsymbol{u}_k{}^H \Delta\boldsymbol{s}}{z_k\,\alpha_k^{(M_k-1)}}\right)^2\right].$$

Substituting the identity $(\Re(z))^2 = \frac{1}{2}(|z|^2 + \Re(z^2))$ into this last equation shows that $\mathrm{var}(\widehat{\delta}_k)$ is asymptotic to

$$\frac{\sigma^2}{2M_k^2}\left(\frac{\boldsymbol{u}_k{}^H\mathbb{E}[\Delta\boldsymbol{s}\Delta\boldsymbol{s}^H]\boldsymbol{u}_k}{\left|z_k\alpha_k^{(M_k-1)}\right|^2} + \Re\left(\frac{\boldsymbol{u}_k{}^H\mathbb{E}[\Delta\boldsymbol{s}\Delta\boldsymbol{s}^T]\boldsymbol{u}_k{}^*}{\left(z_k\alpha_k^{(M_k-1)}\right)^2}\right)\right).$$

However, since $\Delta\boldsymbol{s}$ is a circular complex random vector, $\mathbb{E}[\Delta\boldsymbol{s}\Delta\boldsymbol{s}^T] = \boldsymbol{0}_{N\times N}$. Therefore

$$\mathrm{var}(\widehat{\delta}_k) \sim \frac{\sigma^2}{2M_k^2}\left(\frac{e^{-2\delta_k}}{\left(a_k^{(M_k-1)}\right)^2}\,\boldsymbol{u}_k{}^H\boldsymbol{\Gamma}\,\boldsymbol{u}_k + 0\right),$$

from which equation (16) is derived. Finally, equation (17) can be derived in a similar way, by using the identity $(\Im(z))^2 = \frac{1}{2}(|z|^2 - \Re(z^2))$. ∎

B. Asymptotic performance

The results presented in section IV-B in [4] are proved below. In order to keep the developments concise, it will be supposed that *all* poles are single, although the results remain valid if only the pole of interest is single. Before proving corollaries 7 and 8, we present a lemma used in both proofs.

Lemma 16. *For all $k \in \{0 \dots K-1\}$, the coefficients of the vector $\boldsymbol{u}_k$ defined in equation* (45) *admit the second order expansion*[2]

$$u_k(\tau) = \boldsymbol{1}_{(\tau \geq n-1)}\,\frac{z_k^{t-l+\tau}}{nl} - \boldsymbol{1}_{(\tau \leq l-1)}\,\frac{z_k^{t-l+\tau}}{nl} + O\left(\frac{1}{N^3}\right). \tag{64}$$

Proof: Let $\boldsymbol{V}^n$ be the $n \times r$ Vandermonde matrix introduced in definition 5. The inverse of the matrix $\boldsymbol{Z} = \boldsymbol{V}^{n\,H}\boldsymbol{V}^n$ involved in corollary 11 admits the asymptotic expansion[3] $\boldsymbol{Z}^{-1} = \frac{1}{n}\boldsymbol{I}_r + O\left(\frac{1}{n^2}\right)$. Consequently, the vectors introduced in this corollary verify

$$\begin{aligned}
\boldsymbol{e}_\uparrow &= [1, 0 \dots 0]^T + O\left(\frac{1}{n}\right) &(65)\\
\boldsymbol{e}_\downarrow &= [0 \dots 0, 1]^T + O\left(\frac{1}{n}\right) &(66)\\
\boldsymbol{v}'_\uparrow &= \frac{1}{n}\boldsymbol{v}_\uparrow + O\left(\frac{1}{n^2}\right) &(67)\\
\boldsymbol{v}'_\downarrow &= \frac{1}{n}\boldsymbol{v}_\downarrow + O\left(\frac{1}{n^2}\right). &(68)
\end{aligned}$$

Then, substituting equations (67) and (68) into equation (43), and noticing that the matrix $\boldsymbol{H}_k$ introduced in proposition 1 satisfies $\boldsymbol{H}_k = z_k^{t-l+1}\alpha_k^{(0)}$, yields

$$\begin{aligned}
\boldsymbol{f}_{\downarrow k} &= \frac{z_k^{-t+l-n+1}}{nl}\,\boldsymbol{v}^l(z_k^*) + O\left(\frac{1}{N^3}\right) &(69)\\
\boldsymbol{f}_{\uparrow k} &= \frac{z_k^{-t+l}}{nl}\,\boldsymbol{v}^l(z_k^*) + O\left(\frac{1}{N^3}\right) &(70)
\end{aligned}$$

where $\boldsymbol{v}^l(z) = [1, z \dots z^{l-1}]^T$. Finally, substituting equations (65), (66), (69) and (70) into equation (45) yields expression (64). ∎

Now let us prove corollaries 7 and 8.

Proof of corollary 7: Lemma 16 shows that

$$\begin{aligned}
\boldsymbol{u}_k^H\boldsymbol{u}_k &= \tfrac{2}{n^2 l} + O\left(\tfrac{1}{N^4}\right) \quad \text{if } n \geq l\\
&= \tfrac{2}{nl^2} + O\left(\tfrac{1}{N^4}\right) \quad \text{if } n \leq l
\end{aligned}$$

Equations (21) and (22) are obtained by substituting this result into equations (16) and (17), where $\boldsymbol{\Gamma} = \boldsymbol{I}_N$ (white noise hypothesis). The minimum variance under the constraint $n + l = N + 1$ is reached for $n = 2l = \frac{2}{3}(N+1)$ or for $l = 2n = \frac{2}{3}(N+1)$. ∎

Proof of corollary 8:

First, let us simplify the expression of the matrix $\boldsymbol{B}$ defined in equation (53). Since all poles are supposed to be single, the matrices $\boldsymbol{B}_k$ introduced in equation (54) have dimension $2 \times N$ and are equal to $[0, 1]^T\boldsymbol{u}_k^H$. Besides, $\boldsymbol{V}^N$ is the $N \times K$ Vandermonde matrix, and $\overline{\boldsymbol{V}}^N$ is the corresponding $N \times 2K$ Pascal-Vandermonde matrix. Since the pseudo-inverse of $\boldsymbol{V}^N$ satisfies $\boldsymbol{V}^{N\dagger} = \frac{1}{N}\boldsymbol{V}^{N\,H} + O(\frac{1}{N^2})$, it can be verified that

$$\boldsymbol{V}^{N\dagger}\overline{\boldsymbol{V}}^N\begin{bmatrix}\boldsymbol{B}_0\\ \vdots\\ \boldsymbol{B}_{K-1}\end{bmatrix} = \frac{N}{2}\boldsymbol{J}^H\boldsymbol{U}^H + O\left(\frac{1}{N^2}\right),$$

where $\boldsymbol{J}$ is the Jordan matrix[4] introduced in section II-B in [4] (here the $K \times K$ matrix $\boldsymbol{J}$ is diagonal since all poles are supposed to be single), and $\boldsymbol{U}$ is the $N \times K$ matrix whose columns are the vectors $\boldsymbol{u}_k$. By substitution in equation (53), one obtains

$$\boldsymbol{B}^H = \frac{1}{N}\boldsymbol{V}^{N\,H} - \frac{N}{2}\boldsymbol{J}^H\boldsymbol{U}^H + O\left(\frac{1}{N^2}\right).$$

[2] The function $\boldsymbol{1}_{(.)}$ is one if its argument is true and zero otherwise.

[3] More generally, in presence of multiple poles, the matrix $\boldsymbol{Z}^{-1}$ admits the expansion $\boldsymbol{Z}^{-1} = \frac{1}{n}\boldsymbol{Z}_0^{-1} + O\left(\frac{1}{n^2}\right)$, where the matrix $\boldsymbol{Z}_0^{-1}$ is block-diagonal.

[4] See [5, pp. 121–142] for a definition of Jordan canonical decomposition.

Therefore $\boldsymbol{B}^H\boldsymbol{B}$ is equal to

$$\frac{1}{N^2}\boldsymbol{V}^{N^H}\boldsymbol{V}^N+\frac{N^2}{4}\boldsymbol{J}^H\boldsymbol{U}^H\boldsymbol{U}\boldsymbol{J}-\Re\left(\boldsymbol{V}^{N^H}\boldsymbol{U}\boldsymbol{J}\right)+O\left(\frac{1}{N^2}\right). \tag{71}$$

Besides, $\boldsymbol{V}^{N^H}\boldsymbol{V}^N = N\boldsymbol{I}_r + O(1)$. Moreover, lemma 16 yields

$$\boldsymbol{U}^H\boldsymbol{U} = \frac{2}{\max(n,l)^2\min(n,l)}\boldsymbol{I}_r + O\left(\frac{1}{N^4}\right).$$

Lastly, lemma 16 also shows that the coefficient of indices (k_1, k_2) in the matrix $\boldsymbol{V}^{N^H}\boldsymbol{U}$ is

$$\left(\boldsymbol{V}^{N^H}\boldsymbol{U}\right)_{(k_1,k_2)} = \boldsymbol{v}^N(z_{k_1})^H\boldsymbol{u}_{k_2}$$
$$= \frac{z_{k_2}^{t-l}}{nl}\left(\sum_{\tau=\max(n-1,l)}^{n+l-2}\frac{z_{k_2}}{z_{k_1}}^\tau - \sum_{\tau=0}^{\min(l-1,n-2)}\frac{z_{k_2}}{z_{k_1}}^\tau\right) + O\left(\frac{1}{N^3}\right)$$

(where $\boldsymbol{v}^N(z) = [1, z \dots z^{N-1}]^T$). This last expression is equal to $O\left(\frac{1}{N^3}\right)$ if $z_{k_1} = z_{k_2}$, or $O\left(\frac{1}{N^2}\right)$ if not. Therefore $\Re\left(\boldsymbol{V}^{N^H}\boldsymbol{U}\boldsymbol{J}\right) = O\left(\frac{1}{N^2}\right)$. Finally, equation (71) yields

$$\boldsymbol{B}^H\boldsymbol{B} = \left(\frac{1}{N} + \frac{N^2}{2\max(n,l)^2\min(n,l)}\right)\boldsymbol{I}_r + O\left(\frac{1}{N^2}\right).$$

Equations (23) and (24) are obtained by substituting this result into equations (19) and (20), where $\boldsymbol{\Gamma} = \boldsymbol{I}_N$ (white noise hypothesis). It can be noticed that the first order terms of the diagonal coefficients of $\boldsymbol{B}^H\boldsymbol{B}$ are equal, and that the minimum of their common value under the constraint $n + l = N + 1$ is reached for $n = 2l = \frac{2}{3}(N + 1)$ or for $l = 2n = \frac{2}{3}(N + 1)$. ∎

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