



# **Generalized Wiener filtering for positive alpha-stable random variables**

***Filtrage de Wiener généralisé pour des variables aléatoires positives alpha-stables***

---

Paul Magron  
Roland Badeau  
Antoine Liutkus

**2016D004**

Juin 2016

Département Traitement du Signal et des Images  
Groupe AAO : Audio, Acoustique et Ondes

# Generalized Wiener filtering for positive alpha-stable random variables

## Filtrage de Wiener généralisé pour des variables aléatoires positives alpha-stables

Paul Magron<sup>1</sup>      Roland Badeau<sup>1</sup>      Antoine Liutkus<sup>2</sup>

<sup>1</sup> LTCI, CNRS, Télécom ParisTech, Université Paris-Saclay, 75013, Paris, France  
`<firstname>.<lastname>@telecom-paristech.fr`

<sup>2</sup> Inria, Speech processing team, Villers-lès-Nancy, France  
`<firstname>.<lastname>@inria.fr`

### Abstract

This report provides a mathematical proof of a result which is a generalization of Wiener filtering to Positive  $\alpha$ -stable (PaS) distributions, a particular subclass of the  $\alpha$ -stable distributions family whose support is  $[0; +\infty[$ . PaS distributions are useful to model nonnegative data and since they are heavy-tailed, they present a natural robustness to outliers. In applications such as nonnegative source separation, it is paramount to have a way of estimating the isolated components that constitute a mixture. To address this issue, we derive an estimator of the sources which is given by the conditional expectation of the sources knowing the mixture. It extends the validity of the generalized Wiener filtering to PaS distributions. This allows us to extract the underlying PaS sources from their mixture.

### Key words

Stable distribution, Positive alpha-stable distributions, Wiener filtering, nonnegative source separation

### Résumé

Dans ce rapport, nous fournissons une preuve mathématique d'un résultat qui est une généralisation du filtrage de Wiener appliqué à des variables aléatoires distribuées selon une loi Positive  $\alpha$ -stable (PaS). Ces distributions forment une sous-catégorie des distributions  $\alpha$ -stables de support  $[0; +\infty[$ . Elles sont utilisées pour la modélisation de données non-négatives, et leur queue lourde est un avantage en terme de robustesse aux valeurs aberrantes. Dans des applications telles que la séparation de sources non-négatives, il est primordial de pouvoir estimer les composantes qui constituent un mélange. Dans ce but, nous proposons d'utiliser comme estimateur l'espérance conditionnelle des sources sachant le mélange. Nous montrons ainsi que l'on peut généraliser le filtrage de Wiener en l'appliquant aux variables PaS, ce qui rend possible l'extraction de sources PaS à partir de leur mélange.

### Mots clés

Distribution stable, Distributions positives alpha-stables, filtrage de Wiener, séparation de sources non-négatives

# 1 Introduction

This document features supplementary materials to the reference paper [1]. We consider a nonnegative data matrix  $X$  which is expressed as the sum of  $K$  components  $X_k$ , which follow a Positive  $\alpha$ -stable (P $\alpha$ S) distribution. P $\alpha$ S distributions are a subclass of the  $\alpha$ -stable distributions family, with shape parameter  $\alpha < 1$ , location parameter  $\mu = 0$ , skewness parameter  $\beta = 1$  and scale parameter  $\sigma > 0$ . Their probability density functions are supported by  $\mathbb{R}_+$ , which makes them useful to model nonnegative data. Since they are heavy-tailed, they also present a natural robustness to outliers [2]. In particular, they have been shown appropriate for audio modeling [3]. In this paper, our goal is to prove that an estimator of the P $\alpha$ S-distributed sources  $X_k$  is given by Wiener-like filtering:

$$\hat{X}_k = \frac{\sigma_k^\alpha}{\sum_l \sigma_l^\alpha} \odot X, \quad (1)$$

where  $\odot$  (resp. the fraction bar) denotes the element-wise matrix multiplication (resp. division). A similar result has been demonstrated for Symetric  $\alpha$ -stable (S $\alpha$ S) distributions [4], thus we propose to extend it to P $\alpha$ S distributions. This property is paramount to separate P $\alpha$ S processes from nonnegative mixtures.

## 2 Proof of the P $\alpha$ S Wiener filtering

Following the proof presented in [5], we first demonstrate (1) for  $K = 2$  sources. Then we extend this result to any  $K$  and to matrices.

### 2.1 Case of two P $\alpha$ S variables

Let  $\alpha \in ]0; 1[$ . Let  $s_1$  and  $s_2$  be two independent P $\alpha$ S random variables of scale parameters  $\sigma_1 > 0$  and  $\sigma_2 > 0$  respectively. A commonly used estimator of  $s_1$  is given by  $\hat{s}_1 = \mathbb{E}_{s_1|x}(s_1)$  if this conditional expectation exists. Besides, this expectation do exist if and only if the characteristic function  $\varphi_{s_1|x}(t_1) = \mathbb{E}_{s_1|x}(e^{it_1 s_1})$  is differentiable at  $t_1 = 0$ . If so, then:

$$\hat{s}_1 = \mathbb{E}_{s_1|x}(s_1) = \frac{1}{i} \frac{d\varphi_{s_1|x}}{dt_1}(0). \quad (2)$$

**Step 1: Characteristic function of  $s_1|x$ .** We first determine the characteristic function of  $s_1|x$ . Using the stability property of the P $\alpha$ S distributions,  $x$  also follows a P $\alpha$ S distribution of scale parameter  $\sigma$  such that  $\sigma^\alpha = \sigma_1^\alpha + \sigma_2^\alpha$ . The characteristic function of a P $\alpha$ S distribution is:

$$\forall t_x \in \mathbb{R}, \varphi_x(t_x) = \mathbb{E}_x(e^{it_x x}) = e^{-\sigma^\alpha |t_x|^\alpha + i\Phi \sigma^\alpha |t_x|^\alpha sg(t_x)}, \quad (3)$$

where  $sg(t_x)$  denotes the sign of  $t_x$ , and  $\Phi$  is a constant equal to  $\tan(\frac{\pi\alpha}{2})$  if  $\alpha \neq 1$ . Since  $\alpha \in ]0; 1[$ ,  $\Phi > 0$ . The characteristic function of the random vector  $(s_1, x)$  is:

$$\begin{aligned} \varphi_{s_1, x}(t_1, t_x) &= \mathbb{E}(e^{i(t_1 s_1 + t_x x)}) \\ &= \mathbb{E}(e^{i(t_1 s_1 + t_x (s_1 + s_2))}) \\ &= \mathbb{E}(e^{i((t_1 + t_x) s_1 + t_x s_2)}) \\ &= \varphi_{s_1, s_2}(t_1 + t_x, t_x). \end{aligned}$$

Since  $s_1$  and  $s_2$  are independent,  $\varphi_{s_1, s_2}(t_1 + t_x, t_x) = \varphi_{s_1}(t_1 + t_x) \varphi_{s_2}(t_x)$ . Then:

$$\varphi_{s_1, x}(t_1, t_x) = e^{-\sigma_1^\alpha |t_1 + t_x|^\alpha - \sigma_2^\alpha |t_x|^\alpha + i\Phi(\sigma_1^\alpha |t_1 + t_x|^\alpha sg(t_1 + t_x) + \sigma_2^\alpha |t_x|^\alpha sg(t_x))}. \quad (4)$$

Using [6] (and more precisely eq (5.1.7) p. 226), we obtain the characteristic function of  $s_1|x$ :

$$\varphi_{s_1|x}(t_1) = \frac{\int_{\mathbb{R}} \varphi_{s_1,x}(t_1, t_x) e^{-it_x x} dt_x}{\int_{\mathbb{R}} \varphi_{s_1,x}(0, t_x) e^{-it_x x} dt_x}. \quad (5)$$

**Step 2: Differentiating the characteristic function.** The first-order derivative of the conditional characteristic function is:

$$\frac{d\varphi_{s_1|x}}{dt_1}(t_1) = \frac{\int_{\mathbb{R}} \frac{\partial \varphi_{s_1,x}}{\partial t_1}(t_1, t_x) e^{-it_x x} dt_x}{\int_{\mathbb{R}} \varphi_{s_1,x}(0, t_x) e^{-it_x x} dt_x}, \quad (6)$$

which, applied at  $t_1 = 0$ , leads to:

$$\frac{d\varphi_{s_1|x}}{dt_1}(0) = \frac{\int_{\mathbb{R}} \frac{\partial \varphi_{s_1,x}}{\partial t_1}(0, t_x) e^{-it_x x} dt_x}{\int_{\mathbb{R}} \varphi_{s_1,x}(0, t_x) e^{-it_x x} dt_x}. \quad (7)$$

Note that (6) is well-defined only if it is possible to differentiate under the  $\int$  sign in the numerator of the right member of (5), which we demonstrate below.

In (4) we distinguish two cases:

- If  $t_1 > -t_x$ ,  $\varphi_{s_1,x}(t_1, t_x) = e^{-\sigma_1^\alpha(t_1+t_x)^\alpha - \sigma_2^\alpha|t_x|^\alpha + i\Phi(\sigma_1^\alpha(t_1+t_x)^\alpha + \sigma_2^\alpha|t_x|^\alpha)sg(t_x)}$ , then:

$$\frac{\partial \varphi_{s_1,x}}{\partial t_1}(t_1, t_x) = [-\alpha\sigma_1^\alpha(t_1+t_x)^{\alpha-1} + i\alpha\Phi\sigma_1^\alpha(t_1+t_x)^{\alpha-1}]\varphi_{s_1,x}(t_1, t_x) \quad (8)$$

- If  $t_1 < -t_x$ ,  $\varphi_{s_1,x}(t_1, t_x) = e^{-\sigma_1^\alpha(-t_1-t_x)^\alpha - \sigma_2^\alpha|t_x|^\alpha + i\Phi(-\sigma_1^\alpha(-t_1-t_x)^\alpha + \sigma_2^\alpha|t_x|^\alpha)sg(t_x)}$ , then:

$$\frac{\partial \varphi_{s_1,x}}{\partial t_1}(t_1, t_x) = [\alpha\sigma_1^\alpha(-t_1-t_x)^{\alpha-1} + i\alpha\Phi\sigma_1^\alpha(-t_1-t_x)^{\alpha-1}]\varphi_{s_1,x}(t_1, t_x). \quad (9)$$

In both cases, we obtain the same expression of the first-order derivative:

$$\frac{\partial \varphi_{s_1,x}}{\partial t_1}(t_1, t_x) = \alpha\sigma_1^\alpha[-(t_1+t_x)|t_1+t_x|^{\alpha-2} + i\Phi|t_1+t_x|^{\alpha-1}]\varphi_{s_1,x}(t_1, t_x), \quad (10)$$

which, applied to  $t_1 = 0$ , leads to:

$$\frac{\partial \varphi_{s_1,x}}{\partial t_1}(0, t_x) = \alpha\sigma_1^\alpha[-t_x|t_x|^{\alpha-2} + i\Phi|t_x|^{\alpha-1}]\varphi_{s_1,x}(0, t_x), \quad (11)$$

with

$$\varphi_{s_1,x}(0, t_x) = e^{-\sigma_1^\alpha|t_x|^\alpha - \sigma_2^\alpha|t_x|^\alpha + i\Phi(\sigma_1^\alpha|t_x|^\alpha + \sigma_2^\alpha|t_x|^\alpha)sg(t_x)} \quad (12)$$

$$= e^{-(\sigma_1^\alpha + \sigma_2^\alpha)|t_x|^\alpha + i\Phi(\sigma_1^\alpha + \sigma_2^\alpha)|t_x|^\alpha sg(t_x)}. \quad (13)$$

Let us now demonstrate that (7) is well-defined. To do so, we show that:

$$\frac{\partial \int_{\mathbb{R}_+} \varphi_{s_1,x}(t_1, t_x) e^{-it_x x} dt_x}{\partial t_1}(t_1 = 0) = \int_{\mathbb{R}_+} \frac{\partial \varphi_{s_1,x}}{\partial t_1}(0, t_x) e^{-it_x x} dt_x \quad (14)$$

$$\frac{\partial \int_{\mathbb{R}_-} \varphi_{s_1,x}(t_1, t_x) e^{-it_x x} dt_x}{\partial t_1}(t_1 = 0) = \int_{\mathbb{R}_-} \frac{\partial \varphi_{s_1,x}}{\partial t_1}(0, t_x) e^{-it_x x} dt_x. \quad (15)$$

So let us prove equation (14) (the same proof will hold for (15)). Firstly, we have to upper bound  $\left| \frac{\partial \varphi_{s_1, x}}{\partial t_1}(t_1, t_x) e^{-it_x x} \right|$ . Since  $\alpha \in ]0, 1[$ , equations (4) and (10) yield:

$$\forall t_1 \in \mathbb{R}_+, \forall t_x \in \mathbb{R}_+ \setminus \{0\}, \left| \frac{\partial \varphi_{s_1, x}}{\partial t_1}(t_1, t_x) e^{-it_x x} \right| \leq g(t_1 + t_x) h(t_x) \leq \|g\|_\infty h(t_x), \quad (16)$$

with  $g(t) = \alpha \sigma_1^\alpha \sqrt{1 + \Phi^2} e^{-\sigma_1^\alpha |t|^\alpha} \in L^\infty(\mathbb{R}_+)$  and  $h(t) = \frac{1}{|t|^{1-\alpha}} e^{-\sigma_2^\alpha |t|^\alpha} \in L^1(\mathbb{R}_+)$ . Since the upper bound (16) is independent of  $t_1$  and Lebesgue integrable on  $\mathbb{R}_+ \setminus \{0\}$ , and since  $\{0\}$  is a negligible set, we conclude that

$$\forall t_1 \in \mathbb{R}_+, \frac{\partial \int_{\mathbb{R}_+} \varphi_{s_1, x}(t_1, t_x) e^{-it_x x} dt_x}{\partial t_1} = \int_{\mathbb{R}_+} \frac{\partial \varphi_{s_1, x}}{\partial t_1}(t_1, t_x) e^{-it_x x} dt_x. \quad (17)$$

Therefore (14) is obtained by taking  $t_1 = 0$ . Similarly, we prove equation (15), which concludes the proof of (7).

**Step 3: Integrating the numerator in (7).** It is easy to calculate the first-order derivative of  $\tilde{\varphi}(t_x) = \varphi_{s_1, x}(0, t_x)$ . We can apply the same technique as above (split  $\mathbb{R}$  into its positive and negative parts in order to eliminate the absolute value) to (13) and we obtain:

$$\frac{d\tilde{\varphi}}{dt_x}(t_x) = \alpha(\sigma_1^\alpha + \sigma_2^\alpha)[-t_x |t_x|^{\alpha-2} + i\Phi |t_x|^{\alpha-1}] \varphi_{s_1, x}(0, t_x). \quad (18)$$

By combining (11) and (18), we obtain:

$$\frac{\partial \varphi_{s_1, x}}{\partial t_1}(0, t_x) = \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} \frac{d\tilde{\varphi}}{dt_x}(t_x). \quad (19)$$

Equation (19) is useful since it allows us to calculate the numerator in (7). Indeed,

$$\int_{\mathbb{R}} \frac{\partial \varphi_{s_1, x}}{\partial t_1}(0, t_x) e^{-it_x x} dt_x = \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} \int_{\mathbb{R}} \frac{d\tilde{\varphi}}{dt_x}(t_x) e^{-it_x x} dt_x, \quad (20)$$

and an integration by parts leads to:

$$\int_{\mathbb{R}} \frac{d\tilde{\varphi}}{dt_x}(t_x) e^{-it_x x} dt_x = - \int_{\mathbb{R}} \tilde{\varphi}(t_x) \frac{d(e^{-it_x x})}{dt_x} dt_x = ix \int_{\mathbb{R}} \tilde{\varphi}(t_x) e^{-it_x x} dt_x. \quad (21)$$

Then, combining (7), (20) and (21) leads to:

$$\frac{d\varphi_{s_1|x}}{dt_1}(0) = i \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} x. \quad (22)$$

**Step 4: Obtaining the estimator.** Finally, we combine (22) with (2) in order to obtain the following estimator of  $s_1$ :

$$\hat{s}_1 = \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} x. \quad (23)$$

With the exact same technique, we obtain an estimator for  $s_2$ . Then, if  $K = 2$ , we have the following result:

$$\forall k \in \llbracket 1; K \rrbracket, \hat{s}_k = \frac{\sigma_k^\alpha}{\sum_{l=1}^K \sigma_l^\alpha} x. \quad (24)$$

**Remark:** In fact, this property still holds for any value of  $\beta$  (which is assumed to be equal to 1 here), as long as  $\alpha \neq 1$ . Indeed, under this condition, the characteristic function (3) becomes:

$$\forall t_x \in \mathbb{R}, \varphi_x(t_x) = \mathbb{E}_x(e^{it_x x}) = e^{-\sigma^\alpha |t_x|^\alpha + i\beta \Phi \sigma^\alpha |t_x|^\alpha s g(t_x)}, \quad (25)$$

Thus we simply need to replace the constant  $\Phi$  by  $\beta\Phi$  in the proof to demonstrate this result. The new constant  $\beta\Phi$  can be null for  $\beta = 1$ , but in this case, the distribution becomes Symmetric  $\alpha$ -stable (S $\alpha$ S), and the property still holds, as demonstrated in [4].

## 2.2 Extension to $K$ random variables and matrices

We now prove that (24) still holds for any number of sources  $K$ . Let us consider a sum of  $K \geq 2$  components that follow a P $\alpha$ S distribution of parameter  $\sigma_k$  respectively. Then,  $\forall k \in \llbracket 1; K \rrbracket$ , let  $\tilde{s}_k = \sum_{l \neq k} s_l$ . We have:

$$s_k + \tilde{s}_k = X, \quad (26)$$

and given the additive property of the P $\alpha$ S distributions,  $\tilde{s}_k$  is also P $\alpha$ S-distributed with parameter  $\tilde{\sigma}_k$  such that  $\tilde{\sigma}_k^\alpha = \sum_{l \neq k} \sigma_l^\alpha$ . We can then use (24) with the two sources  $s_k$  and  $\tilde{s}_k$ :

$$\hat{s}_k = \frac{\sigma_k^\alpha}{\sigma_k^\alpha + \tilde{\sigma}_k^\alpha} x = \frac{\sigma_k^\alpha}{\sigma_k^\alpha + \sum_{l \neq k} \sigma_l^\alpha} x = \frac{\sigma_k^\alpha}{\sum_l \sigma_l^\alpha} x. \quad (27)$$

Since this result is valid for each  $k \in \llbracket 1; K \rrbracket$ , then (24) is true for any  $K$ . Finally, this result can be extended to matrices. Since all TF bins are assumed independent, we can apply (24) in each TF bin, which proves (1).

## 3 Conclusion

In this report, we have proved that the use of the Wiener filtering technique, which provides an estimator of the underlying sources composing a mixture, can be extended to Positive  $\alpha$ -stable distributions, according to (1). This result may be highly useful in applications that address the problem of nonnegative source separation.

## References

- [1] P. Magron, R. Badeau, and A. Liutkus, "Separation of nonnegative alpha-stable sources," *IEEE Signal Processing Letters*, 2016, submitted for publication.
- [2] J. P. Nolan, *Stable Distributions - Models for Heavy Tailed Data*. Boston: Birkhauser, 2015, in progress, Chapter 1 online at [academic2.american.edu/~jpnolan](http://academic2.american.edu/~jpnolan).
- [3] N. Bassiou, C. Kotropoulos, and E. Koliopoulou, "Symmetric  $\alpha$ -stable sparse linear regression for musical audio denoising," in *Proc. International Symposium on Image and Signal Processing and Analysis (ISPA)*. Trieste, Italy: IEEE, September 2013, pp. 382–387.
- [4] A. Liutkus and R. Badeau, "Generalized Wiener filtering with fractional power spectrograms," in *Proc. the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, Brisbane, Australia, April 2015.
- [5] R. Badeau and A. Liutkus, "Proof of Wiener-like linear regression of isotropic complex symmetric alpha-stable random variables," *Télecom ParisTech - INRIA Nancy*, Tech. Rep. hal-01069612, September 2014.
- [6] G. Samoradnitsky and M. S. Taqqu, *Stable non-Gaussian random processes: stochastic models with infinite variance*. CRC Press, 1994.



---

**Télécom ParisTech**

Institut Mines-Télécom - membre de ParisTech

46, rue Barrault - 75634 Paris Cedex 13 - Tél. + 33 (0)1 45 81 77 77 - [www.telecom-paristech.fr](http://www.telecom-paristech.fr)

**Département TSI**