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Minimum Sizes of Identifying Codes in Graphs Differing by One Edge or One Vertex

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Résumé

Soit G un graphe simple, non orienté, d'ensemble de sommets V. Pour $v \in V$ et $r \geq 1$, on note $B_{G,r}(v)$ la boule de rayon r et centre v. Un ensemble $\mathcal{C} \subseteq V$ est appelé un *code r-identifiant* dans G si les ensembles $B_{G,r}(v) \cap \mathcal{C}$, $v \in V$, sont tous non vides et distincts. Un graphe G admettant un code r-identifiant est dit *sans r-jumeaux*, et dans ce cas la taille d'un plus petit code r-identifiant dans G est dénotée par $\gamma_r(G)$.

Nous étudions le problème structurel suivant : soit G un graphe sans r-jumeaux, et G^* un graphe obtenu à partir de G en ajoutant ou en retirant un sommet, ou en ajoutant ou en retirant une arête. Si G^* est encore sans r-jumeaux, nous comparons le comportement de $\gamma_r(G)$ et $\gamma_r(G^*)$, et établissons des résultats sur leurs possibles différence et rapport.

Mots clés : Théorie des graphes, Graphes sans jumeaux, Graphes identifiables, Codes identifiants.

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Abstract

Let G be a simple, undirected graph with vertex set V. For $v \in V$ and $r \geq 1$, we denote by $B_{G,r}(v)$ the ball of radius r and centre v. A set $\mathcal{C} \subseteq V$ is said to be an r-*identifying code* in G if the sets $B_{G,r}(v) \cap \mathcal{C}$, $v \in V$, are all nonempty and distinct. A graph G admitting an r-identifying code is called r-*twin-free*, and in this case the size of a smallest r-identifying code in G is denoted by $\gamma_r(G)$.

We study the following structural problem: let G be an r-twin-free graph, and G^* be a graph obtained from G by adding or deleting a vertex, or by adding or deleting an edge. If G^* is still r-twin-free, we compare the behaviours of $\gamma_r(G)$ and $\gamma_r(G^*)$, establishing results on their possible differences and ratios.

Key Words: Graph Theory, Twin-Free Graphs, Identifiable Graphs, Identifying Codes.

1 Foreword

This preprint is a combination of the two submitted articles [9] and [10] which deal with closely related topics. The Introduction and Bibliography are put in common, but Part I, devoted to the addition or deletion of one vertex, and Part II, about the addition or deletion of one edge, have been made so that they can be read independently and in any order; consequently, they have a (very) small intersection, see, e.g., the proofs of Propositions 10, part (a), on page 11 and of Proposition 26, part (a), on page 25. Part I starts at page 7 and Part II at page 18.

2 Introduction

We introduce basic definitions and notation for graphs, for which we refer to, e.g., [1] and [12], and for identifying codes (see [18] and the bibliography at [21]).

We shall denote by $G = (V, E)$ a simple, undirected graph with vertex set V and edge set E, where an *edge* between $x \in V$ and $y \in V$ is indifferently denoted by $\{x, y\}$, $\{y, x\}$, xy or yx. The *order* of a graph is its number of vertices $|V|$.

A *path* $P_n = x_1x_2 \dots x_n$ is a sequence of n distinct vertices x_i, $1 \leq i \leq n$, such that x_ix_{i+1} is an edge for $i \in \{1, 2, \dots, n-1\}$. The *length* of P_n is its number of edges, $n-1$. A *cycle* $C_n = x_1x_2 \dots x_n$ is a sequence of n distinct vertices x_i, $1 \leq i \leq n$, where x_ix_{i+1} is an edge for $i \in \{1, 2, \dots, n-1\}$, and x_nx_1 is also an edge; its length is n.

A graph G is called *connected* if for any two vertices x and y, there is a path between them. It is called *disconnected* otherwise. In a connected graph G, we can define the *distance* between any two vertices x and y, denoted by $d_G(x, y)$, as the length of any shortest path between x and y, since such a path exists. This definition can be extended to disconnected graphs, using the convention that $d_G(x, y) = +\infty$ if there is no path between x and y.

For any vertex $v \in V$ and integer $r \geq 1$, the *ball of radius r and centre v*, denoted by $B_{G,r}(v)$, is the set of vertices within distance r from v:

$$B_{G,r}(v) = \{x \in V : d_G(v, x) \leq r\}.$$

Two vertices x and y such that $B_{G,r}(x) = B_{G,r}(y)$ are called *(G, r)-twins*; if G has no (G, r)-twins, that is, if

$$\forall x, y \in V \text{ with } x \neq y, \;\; B_{G,r}(x) \neq B_{G,r}(y),$$

then we say that G is *r-twin-free*.

Whenever two vertices x and y are within distance r from each other in G, i.e., $x \in B_{G,r}(y)$ and $y \in B_{G,r}(x)$, we say that x and y *r-cover* each other. When three vertices x, y, z are such that $x \in B_{G,r}(z)$ and $y \notin B_{G,r}(z)$, we say that z *r-separates* x and y in G. A set is said to r-separate x and y in G if it contains at least one vertex which does.

A *code* $\mathcal{C}$ is simply a subset of V, and its elements are called *codewords*. For each vertex $v \in V$, the *r-identifying set* of v, with respect to $\mathcal{C}$, is the set of codewords r-covering v, and is denoted by $I_{G,\mathcal{C},r}(v)$:

$$I_{G,\mathcal{C},r}(v) = B_{G,r}(v) \cap \mathcal{C}.$$

We say that $\mathcal{C}$ is an *r-identifying* code [18] if all the sets $I_{G,\mathcal{C},r}(v)$, $v \in V$, are nonempty and distinct: in other words, every vertex is r-covered by at least one codeword, and every pair of vertices is r-separated by at least one codeword.

It is quite easy to observe that a graph G admits an r-identifying code if and only if G is r-twin-free; this is why r-twin-free graphs are also sometimes called *r-identifiable*.

When G is r-twin-free, we denote by $\gamma_r(G)$ the cardinality of a smallest r-identifying code in G. The search for the smallest r-identifying code in given graphs or families of graphs is an important part of the studies devoted to identifying codes.

In this preprint and the forthcoming [9] and [10], we are interested in the following issue: let G be an r-twin-free graph, and G^* be a graph obtained from G by adding or deleting one vertex, or by adding or deleting one edge. Now, if G^* is still r-twin-free, what can be said about $\gamma_r(G)$ compared to $\gamma_r(G^*)$? More specifically, we shall study their difference and, when appropriate, their ratio,

$$\gamma_r(G) - \gamma_r(G^*) \text{ and } \frac{\gamma_r(G)}{\gamma_r(G^*)},$$

as functions of the order of the graph G, and r.

Note that a partial answer to the issue of knowing the conditions for which an r-twin-free graph remains so when one vertex is removed was given in [4] and [6]: any 1-twin-free graph with at least four vertices always possesses at least one vertex whose deletion leaves the graph 1-twin-free; for any $r \geq 1$, any r-twin-free tree with at least $2r + 2$ vertices always possesses at least one vertex whose deletion leaves the graph r-twin-free; on ther other hand, for any $r \geq 3$, there exist r-twin-free graphs such that the deletion of any vertex makes the graph not r-twin-free. The case $r = 2$ remains open.

Of what interest this study is, can be illustrated by the watching of a museum: we place ourselves in the case $r = 1$ and assume that we have to protect a museum, or any other type of premises, using smoke detectors. The museum can be viewed as a graph, where the vertices represent the rooms, and the edges, the doors or corridors between rooms. The detectors are located in some of the rooms and give the alarm whenever there is smoke in their room or in one of the adjacent rooms. If there is smoke in one room and if the detectors are located in rooms corresponding to a 1-identifying code, then, only by knowing which detectors gave the alarm, we can identify the room where someone is smoking.

Of course we want to use as few detectors as possible. Now, what are the consequences, beneficial or not, of closing or opening one room or one door? This is exactly what we investigate here, in the more general case when r can take values other than 1.

In the conclusion of [22], it is already observed, somewhat paradoxically, that a cycle with one vertex less can require more codewords/detectors. In the sequel, we shall exhibit examples of large variations for the minimum size of an identifying code.

A related issue is that of *t-edge-robust* identifying codes, which remain identifying when at most t edges are added or deleted, in any possible way; see, e.g., [15]–[17], [19] or [20].

Let us mention that in the sequel, we shall consider two cases,

(i) both graphs G ands G^* are connected,

(ii) the graph with one edge less or one vertex less may be disconnected, and observe some significant differences. For $r = 1$, this distinction is meaningless, since a vertex v which is linked to all the other vertices guarantees that the graph is connected and does not change anything as far as 1-identification is concerned, in the sense given by the following two easy lemmata.

Lemma 1 *If $G = (V, E)$ is not connected and 1-twin-free, and if G^* is the graph obtained by adding to G a vertex v which is linked to all the vertices in V, then G^* is (connected and) 1-twin-free.*

Proof. First, v is the only vertex in G^* which is connected to every vertex in V. Second, for all vertices $x \in V$, we have $B_{G^*,1}(x) = B_{G,1}(x) \cup \{v\}$, so, by hypothesis on G, these balls are distinct. △

Lemma 2 *If a graph $G = (V, E)$ is 1-twin-free and contains a vertex v which is linked to all the other vertices, then there is an optimal 1-identifying code $\mathcal{C}$ not containing v.*

Proof. Assume that an optimal 1-identifying code $\mathcal{C}$ contains v. Since v cannot 1-separate any pair of vertices in G, its only purpose as a codeword is to 1-cover some vertices not 1-covered by any other codeword; because these vertices are 1-separated by $\mathcal{C}$, only one of them, which we denote by x, can be such that $I_{G,\mathcal{C},1}(x) = \{v\}$. Then $\mathcal{C} \setminus \{v\} \cup \{x\}$ is also optimal and 1-identifying. △

Before we proceed, we still need some additional definitions and notation, we state one theorem on cycles, and we also give two lemmata which, although trivial, will prove useful in the sequel, even implicitly.

For a graph $G = (V, E)$ and a vertex $v \in V$, we denote by G_v the graph with vertex set V' and edge set E', where

$$V' = V \setminus \{v\},\ E' = \{xy \in E : x \in V', y \in V'\}.$$

When we delete the edge $e \in E$ in a graph $G = (V, E)$, we denote the resulting subgraph by $G_e = (V, E_e)$.

In both parts, I and II, we shall use the following result on cycles of even length.

Theorem 3 *[3] For all $r \geq 1$ and for all even n, $n \geq 2r + 4$, we have:*

$$\gamma_r(C_n) = \frac{n}{2}.$$

$\triangle$

If $G = (V, E)$ is a graph and $\mathcal{S}$ is a subset of V, we say that two vertices $x \in V$ and $y \in V$ are $(G, \mathcal{S}, r)$-*twins* if

$$I_{G,\mathcal{S},r}(x) = I_{G,\mathcal{S},r}(y).$$

In other words, x and y are not r-separated by $\mathcal{S}$ in G. By definition, if $\mathcal{C}$ is r-identifying in G, then no $(G, \mathcal{C}, r)$-twins exist.

Lemma 4 *[$(G, \mathcal{S}, r)$-twin transitivity] In a graph $G = (V, E)$, if x, y, z are three distinct vertices, if $\mathcal{S}$ is a subset of V, if x and y are $(G, \mathcal{S}, r)$-twins and if y and z are $(G, \mathcal{S}, r)$-twins, then x and z are $(G, \mathcal{S}, r)$-twins.* $\triangle$

Lemma 5 *If $\mathcal{C}$ is an r-identifying code in a graph $G = (V, E)$, then so is any set $\mathcal{S}$ such that*

$$\mathcal{C} \subseteq \mathcal{S} \subseteq V.$$

$\triangle$

Part I: Addition and deletion of one vertex

We present the main six results of this part in the following way. In Section 3 we consider the case $r = 1$: we study how *large* $\gamma_1(G_x) - \gamma_1(G)$ can be (Proposition 6), then Theorem 7 states exactly how *small* this difference can be (namely, -1). In Section 4, we study how *small* $\gamma_r(G_x) - \gamma_r(G)$ can be for any $r \geq 2$ (Proposition 10) and it so happens that the graphs we use are connected; then we study how *large* this difference can be, in the following three cases: (i) $r \geq 2$, r is even and the graphs are connected (Proposition 12 in Section 4.1); (ii) $r \geq 3$, r is odd and the graphs are connected (Proposition 14 in Section 4.2); (iii) $r \geq 2$ and the graph G_x is disconnected (Proposition 16 in Section 4.3). In all these sections, the number n represents the order of either G or G_x. A general conclusion recapitulates our results.

3 The case $r = 1$

Proposition 6 *There exist two (connected) 1-twin-free graphs G and G_x, with $n = 4\beta + 6$ and $4\beta + 5$ vertices respectively, such that $\gamma_1(G) = 2\beta + 3$ and $\gamma_1(G_x) = 3\beta + 3$, where β is any integer greater than or equal to 1.*

Remark preceding the proof. The difference $\gamma_1(G_x) - \gamma_1(G) = \beta$ can be made arbitrarily large; in terms of n, the order of G, we can see that we have:

$$\gamma_1(G_x) - \gamma_1(G) = \frac{n-6}{4}. \tag{1}$$

We can also consider the ratio between $\gamma_1(G_x)$ and $\gamma_1(G)$:

$$\frac{\gamma_1(G_x)}{\gamma_1(G)} = \frac{3n-6}{2n}, \tag{2}$$

which is equivalent to $\frac{3}{2}$ when β, or n, goes to infinity. An open question is to know whether these difference or ratio can be made larger.

On the other hand, we shall see in Theorem 7 that any two 1-twin-free graphs H and H_x must satisfy $\gamma_1(H) - \gamma_1(H_x) \leq 1$.

Proof of Proposition 6. Let $\beta \geq 1$ be an integer.

For $1 \leq i \leq \beta + 1$, we consider copies (V_i, E_i) of the path P_4, with the following notation:

$$V_i = \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}\}, \text{ and } E_i = \{\{v_{i,j}, v_{i,j+1}\} : 1 \leq j \leq 3\}.$$

Let $G = (V, E)$, where

$$V = \cup_{1 \leq i \leq \beta+1} V_i \cup \{x\} \cup \{y\} \quad \text{and}$$

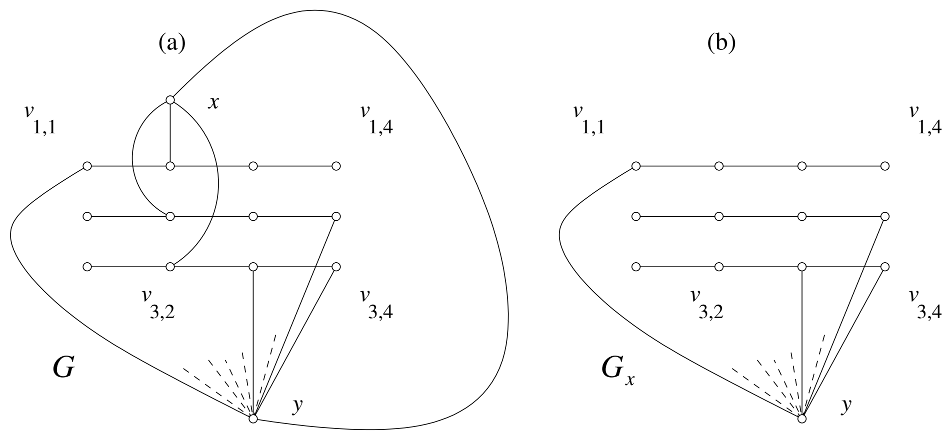

Figure 1: The graphs G and G_x, for $\beta = 2$, in Proposition 6.

$$E = \cup_{1 \le i \le \beta+1} E_i \cup \{\{y, s\} : s \in V \setminus \{y\}\} \cup \{\{x, v_{i,2}\} : 1 \le i \le \beta + 1\},$$

see Figure 1(a). Note that the vertex y is used only to make sure that G_x is connected, which is not strictly necessary; the same construction could have been done without y, with G a tree and G_x a forest.

Then G has $n = 4\beta + 6$ vertices, and we claim that: (a) $\gamma_1(G) = 2\beta + 3$, and (b) $\gamma_1(G_x) = 3\beta + 3$, from which (1) and (2) follow.

Proof of (a). The code

$$\{x\} \cup \{v_{i,2}, v_{i,3} : 1 \le i \le \beta + 1\}$$

is clearly 1-identifying in G. To establish the lower bound, we use Lemma 2 and assume that $\mathcal{C}$ is an optimal 1-identifying code not containing y. Then we need at least two codewords on each copy of P_4, because each end must be 1-covered by at least one codeword. Finally, since $\gamma_1(P_4) = 3$, it is straightforward to see that the best we can do is to take advantage of x, take it as a codeword, and have $2\beta + 3$ codewords.

Proof of (b). Thanks to Lemma 2, this is obvious, using that $\gamma_1(P_4) = 3$. △

Note that, here and also in Proposition 10, we could have contented ourselves with inequalities, here $\gamma_1(G) \le 2\beta + 3$ and $\gamma_1(G_x) \ge 3\beta + 3$, so as to obtain

$$\gamma_1(G_x) - \gamma_1(G) \ge \beta = \frac{n-6}{4} \quad \text{and} \quad \frac{\gamma_1(G_x)}{\gamma_1(G)} \ge \frac{3\beta+3}{2\beta+3} = \frac{3n-6}{2n}.$$

Theorem 7 *Let $G = (V, E)$ be any 1-twin-free graph with at least three vertices. For any vertex $x \in V$ such that G_x is 1-twin-free, we have:*

$$\gamma_1(G_x) \ge \gamma_1(G) - 1. \tag{3}$$

Proof. Cf. [13, Prop. 3]. For completeness, we still give a proof. Let $x \in V$ be such that G_x is 1-twin-free. Let $\mathcal{C}_x$ be a minimum 1-identifying code in G_x: $|\mathcal{C}_x| = \gamma_1(G_x)$. There are two cases: either (a) x is not 1-covered (in G) by any codeword of $\mathcal{C}_x$, or (b) x is 1-covered (in G) by at least one codeword of $\mathcal{C}_x$.

(a) In this case, let $\mathcal{C} = \mathcal{C}_x \cup \{x\}$. Then $\mathcal{C}$ is clearly 1-identifying in G (in particular, thanks to Lemma 5); therefore, $\gamma_1(G) \leq \gamma_1(G_x) + 1$.

(b) x is 1-covered by $y \in \mathcal{C}_x$. If $\mathcal{C}_x$ is 1-identifying in G, then $\gamma_1(G) \leq \gamma_1(G_x)$, and we are done. So we assume that $\mathcal{C}_x$ is not 1-identifying in G. This means that either (i) at least one vertex in G is not 1-covered by $\mathcal{C}_x$, or (ii) at least two vertices in G are not 1-separated by $\mathcal{C}_x$.

(i) Since $\mathcal{C}_x$ 1-covers any vertex in G_x and x is linked to $y \in \mathcal{C}_x$, this case is impossible.

(ii) Let $u, v \in V$ be two distinct vertices which are not 1-separated by $\mathcal{C}_x$. One of them is necessarily x, and without loss of generality, we assume that $x = u$.

Now, v is unique by Lemma 4: $\mathcal{C}_x$ is not 1-identifying in G only because *one* pair of vertices, x and v, is not 1-separated by $\mathcal{C}_x$.

Since G is 1-twin-free, there is a vertex z which 1-covers exactly one of the vertices v and x. We set $\mathcal{C} = \mathcal{C}_x \cup \{z\}$, and we obtain a 1-identifying code in G, so $\gamma_1(G) \leq \gamma_1(G_x) + 1$. △

Corollary 8 *If* $\gamma_1(G_x) \leq a$ *and* $\gamma_1(G) \geq a + 1$*, then* $\gamma_1(G_x) = a$ *and* $\gamma_1(G) = a + 1$. △

Note that we made no assumption on the connectivity of G or G_x. Examples where $\gamma_1(G_x) = \gamma_1(G) - 1$, or $\gamma_1(G_x) = \gamma_1(G)$, are numerous and easy to find.

Conclusion 9 *Provided that the graphs considered are* 1*-twin-free, we can see, using Proposition 6 and Theorem 7, that* $\gamma_1(G_x) - \gamma_1(G)$ *cannot be smaller than* -1*, but examples exist where it can be as large as, approximately,* $\frac{n}{4}$*, and where the ratio* $\frac{\gamma_1(G_x)}{\gamma_1(G)}$ *can be as large as, approximately,* $\frac{3}{2}$.

4 The case $r \geq 2$

Things are different for $r \geq 2$, since we can exhibit pairs of graphs (G, G_x) proving that $\gamma_r(G) - \gamma_r(G_x)$ can be arbitrarily large or small.

We first give a result with $\gamma_r(G) - \gamma_r(G_x)$ arbitrarily large. Note that we obtain this result with connected graphs: we found no better with disconnected graphs.

Proposition 10 *There exist two (connected)* r*-twin-free graphs* G_x *and* G*, with* $n = pr + 1$ *and* $pr + 2$ *vertices respectively, such that*

$$\gamma_r(G_x) = p + 2r - 3 = \frac{n + 2r^2 - 3r - 1}{r} \quad \textit{and} \quad \gamma_r(G) = r(p-1) + 1 = n - r,$$

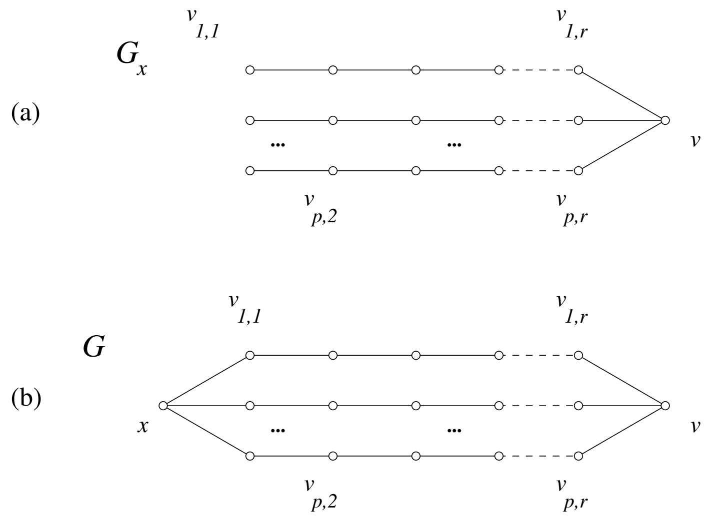

Figure 2: The graphs G_x and G in Proposition 10.

where p is any integer greater than or equal to 3.

Remark preceding the proof. The difference

$$\gamma_r(G) - \gamma_r(G_x) = p(r-1) - 3r + 4$$

can be made arbitrarily large; in terms of n, the number of vertices of G_x, we can see that we have:

$$\gamma_r(G) - \gamma_r(G_x) = \frac{(n-3r)(r-1)+1}{r}, \tag{4}$$

which is equivalent to $\frac{n(r-1)}{r}$ when r is fixed and n goes to infinity. Also:

$$\frac{\gamma_r(G)}{\gamma_r(G_x)} = \frac{r(n-r)}{n+2r^2-3r-1}, \tag{5}$$

which is equivalent to r when we increase n. Again, is it possible to improve on (4) or (5)?

Proof of Proposition 10. Let $r \geq 2$ and $p \geq 3$ be integers; we put the cart before the horse and, before defining G, we describe G_x in the following informal way, partially illustrated in Figure 2(a): G_x consists of p copies of the path P_r, and in each copy the last vertex is linked to v. This graph has $n = pr + 1$ vertices. Next, we construct the graph G consisting of G_x to which we add one vertex x, linked to each first vertex of all the copies of P_r. See Figure 2(b). We claim that: (a) $\gamma_r(G_x) = p + 2r - 3$, and (b) $\gamma_r(G) = r(p-1) + 1$, from which (4) and (5) follow.

Proof of (a). The code

$$\mathcal{C} = \{v_{1,i} : 2 \le i \le r\} \cup \{v_{2,i} : 1 \le i \le r\} \cup \{v_{j,1} : 3 \le j \le p\}, \qquad (6)$$

i.e., the code consisting of all the vertices of the first two copies of P_r, except $v_{1,1}$, and the first vertex of each of the following copies, is r-identifying in G_x; this it is straightforward to check. So $\gamma_r(G_x) \le (r-1)+r+(p-2) = p+2r-3$. We now prove that $\gamma_r(G_x) \ge p+2r-3$. The following two observations will be useful. For $1 \le i \le p$ and $2 \le k \le r$, we have:

$$B_{G_x,r}(v_{i,r-k+1}) \Delta B_{G_x,r}(v_{i,r-k+2}) = \{v_{j,k} : 1 \le j \le p, j \ne i\}, \qquad (7)$$

where Δ stands for the symmetric difference, and for $1 \le i < j \le p$:

$$B_{G_x,r}(v_{i,r}) \Delta B_{G_x,r}(v_{j,r}) = \{v_{i,1}, v_{j,1}\}. \qquad (8)$$

The consequences are immediate. First, in order to have the vertices $v_{i,r}$, $1 \le i \le p$, pairwise r-separated in G_x, we see by (8) that we need at least $p-1$ codewords among the p vertices $v_{i,1}$; second, for k fixed between 2 and r, we see, using (7), that we need at least two codewords among the p vertices $v_{i,k}$. So $\gamma_r(G_x) \ge (p-1)+2(r-1) = p+2r-3$, and Claim (a) is proved.

Proof of (b). Note that in G, for i and j such that $1 \le i < j \le p$, the set of vertices

$$\{x\} \cup \{v_{i,k} : 1 \le k \le r\} \cup \{v\} \cup \{v_{j,k} : 1 \le k \le r\}$$

forms the cycle C_{2r+2}, which is r-twin-free and is denoted by $C(i,j)$. On such a cycle, we say that the vertex z is the *opposite* of the vertex y if z is the (only) vertex at distance $r+1$ from y.

We claim that, for k fixed between 1 and r, among the p vertices $v_{i,k}$, at least $p-1$ of them belong to any r-identifying code $\mathcal{C}$ in G. Indeed, assume on the contrary that two vertices, say $v_{1,k}$ and $v_{2,k}$, are not in $\mathcal{C}$; then their opposite vertices in $C(1,2)$, $v_{2,r-k+1}$ and $v_{1,r-k+1}$ respectively, cannot be r-separated by $\mathcal{C}$.

Finally, the fact that $B_{G,r}(v) \Delta B_{G,r}(x) = \{v, x\}$ shows that v or x belong to $\mathcal{C}$, and finally $\gamma_r(G) \ge (p-1)r+1$. On the other hand,

$$\{v\} \cup \{v_{i,k} : 2 \le i \le p, 1 \le k \le r\}$$

is an r-identifying code in G, with size $(p-1)r+1$, thus Claim (b) is proved. Note that this code contains all the vertices in G, except the $r+1$ vertices x and $v_{1,k}$, $1 \le k \le r$. $\triangle$

Conclusion 11 *When $r \ge 2$, Proposition 10 provides pairs of graphs proving that $\gamma_r(G_x) - \gamma_r(G)$ can be, asymptotically, as small as approximately $-\frac{n(r-1)}{r}$, and $\frac{\gamma_r(G_x)}{\gamma_r(G)}$ can be, asymptotically, as small as approximately $\frac{1}{r}$, and this can even be obtained with connected examples.*

Then we turn to examples where $\gamma_r(G_x) - \gamma_r(G)$ is arbitrarily large. We start with connected graphs, and have two subcases, r even and r odd.

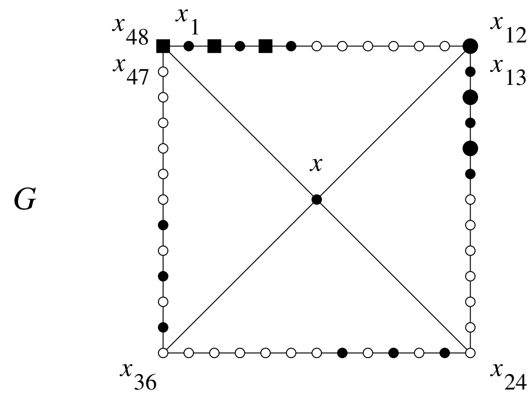

Figure 3: Graph G in Proposition 12, for $r = 6$ and $k = 4$. Squares and circles, white or black, small or large, are vertices. The 19 black vertices constitute a 6-identifying code in G.

4.1 Case of a connected graph G_x and $r \geq 2$, r even

Proposition 12 *There exist two (connected) r-twin-free graphs G and G_x, with $n+1$ and n vertices respectively, such that*

$$\gamma_r(G_x) - \gamma_r(G) \geq \frac{n}{4} - (r+1), \tag{9}$$

$$\frac{\gamma_r(G_x)}{\gamma_r(G)} \geq \frac{2n}{n+4r+4}. \tag{10}$$

Remark preceding the proof. The two lower bounds (9) and (10) are equivalent to $n/4$ and 2 when n increases with respect to r. An open question is whether these two inequalities can be improved.

Proof of Proposition 12. Let $r \geq 2$ be an even integer, and n be an (even) integer such that $n = k \cdot 2r$, $k \geq 2$; let $G_x = C_n = x_1 x_2 \dots x_n$ be the cycle of length n and G be the graph obtained from G_x by adding the vertex x and linking it to the k vertices $x_{j \cdot 2r}$, $1 \leq j \leq k$. See Figure 3, which illustrates the case $r = 6, k = 4, n = 48$ and G has 49 vertices.

We know by Theorem 3 that $\gamma_r(G_x) = \frac{n}{2}$, and we claim that

$$\gamma_r(G) \leq 1 + (k+2)\frac{n}{4k} = \frac{n}{4} + r + 1,$$

from which (9) and (10) follow. Proving this claim, by exhibiting an r-identifying code for G, is tedious and of no special interest; therefore, we content ourselves with showing how it works in the case $r = 6$, $n = 48$, hoping that this will help the reader to gain an insight into the general case. We consider a first set

$$\mathcal{S} = \{x, x_1, x_3, x_5, x_{13}, x_{15}, x_{17}, x_{25}, x_{27}, x_{29}, x_{37}, x_{39}, x_{41}\},$$

see the small black circles in Figure 3. It is now quite straightforward to observe that the pairs $\{x_{48}, x_1\}$, $\{x_2, x_3\}$ and $\{x_4, x_5\}$ are pairs of $(G, \mathcal{S}, 6)$-twins, as well as $\{x_{12}, x_{13}\}$, $\{x_{14}, x_{15}\}$, $\{x_{16}, x_{17}\}$, $\{x_{24}, x_{25}\}$, $\{x_{26}, x_{27}\}$, $\{x_{28}, x_{29}\}$, $\{x_{36}, x_{37}\}$, $\{x_{38}, x_{39}\}$ and $\{x_{40}, x_{41}\}$, for reasons of symmetry, and that they are the only ones.

Let us consider the first three pairs, $\{x_{48}, x_1\}$, $\{x_2, x_3\}$, $\{x_4, x_5\}$. Using edges going through x, they can be 6-separated, for instance, by the vertices x_{16}, x_{14} and x_{12} (see the large black circles), and these three vertices also 6-separate the other pairs of $(G, \mathcal{S}, 6)$-twins, except for $\{x_{12}, x_{13}\}$, $\{x_{14}, x_{15}\}$, $\{x_{16}, x_{17}\}$. These three pairs can however be 6-separated by three more codewords, for instance x_4, x_2 and x_{48}, see the black squares in Figure 3. Now the code

$$\mathcal{C} = \mathcal{S} \cup \{x_{12}, x_{14}, x_{16}, x_{48}, x_2, x_4\}$$

is 6-identifying in G and has $1 + (4 \times 3) + (2 \times 3) = 19$ codewords.

In the general case,

$$\mathcal{S} = \{x\} \cup \{x_{1+j\cdot 2r}, x_{3+j\cdot 2r}, \ldots, x_{r-1+j\cdot 2r} : 0 \leq j \leq k-1\},$$

there are $k \times \frac{r}{2}$ pairs of $(G, \mathcal{S}, r)$-twins, and $\mathcal{C}$ can be chosen, for instance, as

$$\mathcal{C} = \mathcal{S} \cup \{x_n, x_2, \ldots, x_{r-2}\} \cup \{x_{2r}, x_{2r+2}, \ldots, x_{2r+(r-2)}\},$$

which shows that the cardinality of $\mathcal{C}$ is

$$1 + (k \times \frac{r}{2}) + (2 \times \frac{r}{2}) = 1 + (k+2)\frac{n}{4k},$$

and so $\gamma_r(G) \leq 1 + (k+2)\frac{n}{4k}$. △

Conclusion 13 *When r is even, Proposition 12 gives pairs of connected graphs proving that $\gamma_r(G_x) - \gamma_r(G)$ can be, asymptotically, as large as approximately $\frac{n}{4}$, and $\frac{\gamma_r(G_x)}{\gamma_r(G)}$ can be, asymptotically, as large as approximately 2.*

4.2 Case of a connected graph G_x and $r \geq 3$, r odd

Proposition 14 *There exist two (connected) r-twin-free graphs G and G_x, with $n+1$ and n vertices respectively, such that*

$$\gamma_r(G_x) - \gamma_r(G) \geq \frac{n(3r-1)}{12r} - r, \tag{11}$$

$$\frac{\gamma_r(G_x)}{\gamma_r(G)} \geq \frac{6nr}{n(3r+1) + 12r^2}. \tag{12}$$

Remark preceding the proof. An open question is to know whether these two inequalities can be improved. The two lower bounds are equivalent to $\frac{n(3r-1)}{12r}$ and $\frac{6r}{3r+1}$ when r is fixed and n goes to infinity.

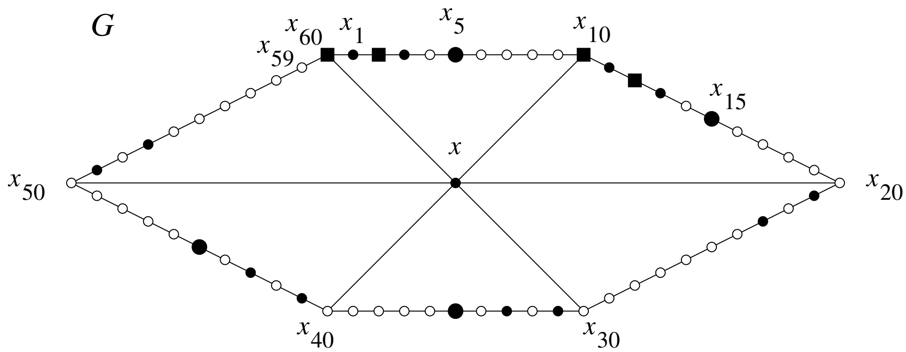

Figure 4: Graph G in Proposition 14, for $r = 5$ and $k = 6$. Squares and circles, white or black, small or large, are vertices. The 21 black vertices constitute a 5-identifying code in G.

Proof of Proposition 14. Let $r \geq 3$ be an odd integer, and n be an (even) integer such that $n = k \cdot 2r$, where $k \geq 3$ is a multiple of 3; let $G_x = C_n = x_1 x_2 \dots x_n$ be the cycle of length n and G be the graph obtained from G_x by adding the vertex x and linking it to the k vertices $x_{j \cdot 2r}$, $1 \leq j \leq k$. See Figure 4, which illustrates the case $r = 5, k = 6, n = 60$ and G has 61 vertices.

We know by Theorem 3 that $\gamma_r(G_x) = \frac{n}{2}$, and we claim that

$$\gamma_r(G) \leq \frac{n}{4} + \frac{n}{12r} + r,$$

from which (11) and (12) follow. Again, proving this claim is of no interest here, and we just show how it works in the case $r = 5$, $n = 60$. We consider a first set

$$\mathcal{S} = \{x, x_1, x_3, x_{11}, x_{13}, x_{21}, x_{23}, x_{31}, x_{33}, x_{41}, x_{43}, x_{51}, x_{53}\},$$

see the small black circles in Figure 4. It is straightforward to see that only the following sets of $(G, \mathcal{S}, 5)$-twins exist:

- (i) $\{x, x_{10}, x_{20}, x_{30}, x_{40}, x_{50}, x_{60}\}$,
- (ii) $\{x_{59}, x_1, x_2\}$ together with the five symmetrical sets $\{x_9, x_{11}, x_{12}\}, \dots$,
- (iii) $\{x_3, x_4\}$ together with the five symmetrical sets $\{x_{13}, x_{14}\}, \dots$

The first two cases are annoying and will be "expensive" because they present symmetries with respect to x. Define the set $\mathcal{T}$ as follows:

$$\mathcal{T} = \mathcal{S} \cup \{x_5, x_{15}, x_{35}, x_{45}\},$$

see the large black circles in Figure 4. Now in Case (i), all the vertices are 5-separated by the vertices in $\mathcal{T} \setminus \mathcal{S}$, and so are x_{59} on the one hand and x_1, x_2 on the other hand, as well as their symmetrical counterparts from

Case (ii). The remaining pairs of $(G, \mathcal{T}, 5)$-twins are $\{x_1, x_2\}$, $\{x_3, x_4\}$ and the 10 pairs obtained by symmetry. As in the proof of Proposition 12, these handle very economically: the vertex x_{60} 5-separates the 5 pairs $\{x_{13}, x_{14}\}$, $\ldots$, $\{x_{53}, x_{54}\}$, and so does x_2 for $\{x_{11}, x_{12}\}$, $\ldots$, $\{x_{51}, x_{52}\}$; finally, $\{x_1, x_2\}$ and $\{x_3, x_4\}$ can be 5-separated, for instance, by x_{10} and x_{12}, see the black squares in Figure 4:

$$\mathcal{C} = \mathcal{T} \cup \{x_{60}, x_2, x_{10}, x_{12}\}$$

is a 5-identifying code in G and has $1 + (6 \times 2) + (4 \times 1) + (2 \times 2) = 21$ codewords. In the general case,

$$\mathcal{S} = \{x\} \cup \{x_{1+j\cdot 2r}, x_{3+j\cdot 2r}, \ldots, x_{r-2+j\cdot 2r} : 0 \le j \le k-1\}$$

contains $1 + (k \times \frac{r-1}{2})$ vertices; then

$$\mathcal{T} = \mathcal{S} \cup \{x_{r+j\cdot 2r} : 0 \le j \le k-1, j \text{ not congruent to 2 modulo 3}\}$$

contains $|\mathcal{S}| + \frac{2k}{3}$ elements, and finally we take

$$\mathcal{C} = \mathcal{T} \cup \{x_n, x_2, \ldots, x_{r-3}\} \cup \{x_{2r}, x_{2r+2}, \ldots, x_{2r+(r-3)}\},$$

which shows that

$$\gamma_r(G) \le 1 + (k \times \frac{r-1}{2}) + \frac{2k}{3} + (2 \times \frac{r-1}{2}) = \frac{n}{4} + \frac{n}{12r} + r.$$

$\triangle$

Conclusion 15 *When $r \ge 3$ and r is odd, Proposition 14 gives pairs of connected graphs proving that $\gamma_r(G_x) - \gamma_r(G)$ can be, asymptotically, as large as approximately $\frac{n(3r-1)}{12r}$, and $\frac{\gamma_r(G_x)}{\gamma_r(G)}$ can be, asymptotically, as large as approximately $\frac{6r}{3r+1}$.*

If we do not require to consider a connected graph G_x, then we can obtain a much larger difference or ratio than in (9)-(12), we need consider only one case, whatever the parity of r is, and moreover the construction is easy to understand; see next section.

4.3 Case of a disconnected graph G_x and $r \ge 2$, r even or odd

Proposition 16 *There exist two graphs G and G_x, with $p(2r+1)+1$ and $n = p(2r+1)$ vertices respectively, such that*

$$\gamma_r(G_x) - \gamma_r(G) \ge \frac{n(2r-2)}{2r+1} - 2r, \tag{13}$$

$$\frac{\gamma_r(G_x)}{\gamma_r(G)} \ge \frac{nr}{n + 4r^2 + 2r}. \tag{14}$$

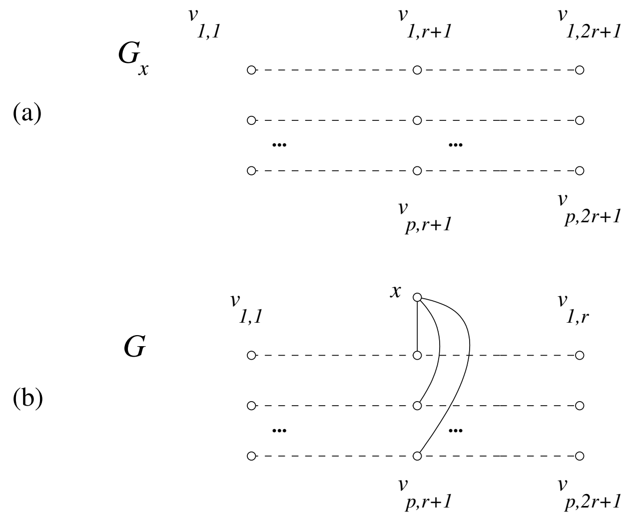

Figure 5: The graphs G_x and G in Proposition 16.

Remark preceding the proof. These two lower bounds are equivalent to $\frac{n(2r-2)}{2r+1}$ and r when n increases. Can they be improved?

Proof of Proposition 16. Let $r \geq 2$ and $p \geq 3$ be integers; the graph G_x consists of p copies of the path P_{2r+1}, and G is obtained by adding the vertex x and linking it to all the middle vertices of the path copies, see Figure 5. We claim that: (a) $\gamma_r(G_x) = 2pr$ and (b) $\gamma_r(G) \leq 2p + 2r$, from which (13) and (14) follow.

Proof of (a). The result comes from the obvious fact that $\gamma_r(P_{2r+1}) = 2r$.
Proof of (b). It is not difficult to check that

$$\mathcal{C} = \{x\} \cup \{v_{i,1}, v_{i,2r+1} : 1 \leq i \leq p-1\} \cup \{v_{p,j} : 1 \leq j \leq 2r+1\}$$

is indeed r-identifying in G. Note however that, for simplicity, we chose to give the bound $2p + 2r$, when actually, with a little more care, $2p + 2r - 3$ can be reached, which would improve only slightly on (13) and (14). △

Conclusion 17 *Proposition 16 gives pairs of graphs (G, G_x), where G_x is not connected, proving that $\gamma_r(G_x) - \gamma_r(G)$ can be, asymptotically, as large as approximately $\frac{n(2r-2)}{2r+1}$, and $\frac{\gamma_r(G_x)}{\gamma_r(G)}$ can be, asymptotically, as large as approximately r.*

5 General conclusion

Table 1 recapitulates the results obtained in the previous sections, using in particular the partial conclusions 9, 11, 13, 15 and 17 at the end of each section; these are stated for n large with respect to r, so it doesn't matter to know if n is the order of G or of G_x. We only consider the difference $\gamma_r(G_x) - \gamma_r(G)$ and the ratio $\frac{\gamma_r(G_x)}{\gamma_r(G)}$.

r	r	comment	$\gamma_r(G_x) - \gamma_r(G)$	$\frac{\gamma_r(G_x)}{\gamma_r(G)}$	reference
$= 1$		impossible to have	< -1		Concl. 9
$= 1$		connected graphs	$\gtrapprox \frac{n}{4}$	$\gtrapprox 3/2$	Concl. 9
≥ 2	any	(connected) graphs	$\lessapprox -\frac{n(r-1)}{r}$	$\lessapprox 1/r$	Concl. 11
≥ 2	even	connected graphs	$\gtrapprox \frac{n}{4}$	$\gtrapprox 2$	Concl. 13
≥ 2	odd	connected graphs	$\gtrapprox \frac{n(3r-1)}{12r}$	$\gtrapprox \frac{6r}{3r+1}$	Concl. 15
≥ 2	any	graphs	$\gtrapprox \frac{n(2r-2)}{2r+1}$	$\gtrapprox r$	Concl. 17

Table 1: The difference $\gamma_r(G_x) - \gamma_r(G)$ and ratio $\frac{\gamma_r(G_x)}{\gamma_r(G)}$, as functions of n and r.

Part II: Addition and deletion of one edge

This part is organized as follows. Section 6 is devoted to the case $r = 1$; here, the difference $\gamma_1(G_e) - \gamma_1(G)$ must lie between -2 and $+2$. Then in the beginning of Section 7, we study how *small* $\gamma_r(G_e) - \gamma_r(G)$ can be for any $r \geq 2$, and it so happens that the graphs we use are connected; then we study how *large* this difference can be, when G_e is connected (Section 7.1) and when G_e is not connected (Section 7.2); in these three cases with $r \geq 2$, we obtain bounds on $\gamma_r(G_e) - \gamma_r(G)$ depending on r, namely $-r$, $r - 1$ and $2r - 3$. A conclusion recapitulates our results.

6 The case $r = 1$

The difference $\gamma_1(G_e) - \gamma_1(G)$ can vary only inside the set $\{-2, -1, 0, 1, 2\}$ (Theorem 22), and these five values can be reached (Examples 19, 21 and 23).

We first study how small $\gamma_1(G_e) - \gamma_1(G)$ can be. Putting the cart before the horse, in the next theorem we first define G_e, and only then, G.

Theorem 18 *Let $G_e = (V, E_e)$ be a 1-twin-free graph with at least four vertices, let x and y be two distinct vertices in V such that $e = xy \notin E_e$, and let $G = (V, E)$ with $E = E_e \cup \{xy\}$. Assume that G is also 1-twin-free.*

If $\mathcal{C}_e$ is a 1-identifying code in G_e, then there exists a 1-identifying code $\mathcal{C}$ in G with

$$|\mathcal{C}| \leq |\mathcal{C}_e| + 2.$$

As a consequence, we have:

$$\gamma_1(G_e) - \gamma_1(G) \geq -2. \tag{15}$$

Proof. Since we add an edge when going from G_e to G, all vertices remain 1-covered, in G, by at least one codeword in $\mathcal{C}_e$.

Since we only add the edge xy, only the balls of x and y are modified in G. As a consequence, only the following pairs are possible $(G, \mathcal{C}_e, 1)$-twins:

- x and y,
- x and u with $u \neq x, u \neq y$,
- y and v with $v \neq x, v \neq y$.

Moreover, x and u', with $u' \neq u, u' \neq x, u' \neq y$, cannot be $(G, \mathcal{C}_e, 1)$-twins since this would imply, by Lemma 4, that u and u' are $(G, \mathcal{C}_e, 1)$-twins, hence $(G_e, \mathcal{C}_e, 1)$-twins, which would contradict the fact that $\mathcal{C}_e$ is 1-identifying in G_e. The same is true for y and v', with $v' \neq v, v' \neq x, v' \neq y$. So at most three pairs of $(G, \mathcal{C}_e, 1)$-twins can appear.

Similarly, if these three pairs of $(G, \mathcal{C}_e, 1)$-twins all do appear, then u and v are $(G, \mathcal{C}_e, 1)$-twins, which leads to the same contradiction, *unless* $u = v$. In this case, because G is 1-twin-free, we can pick an additional

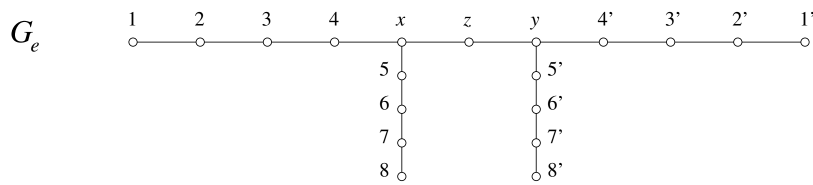

Figure 6: Graph G_e in Example 19.

codeword c_1 1-separating x and u by, say, 1-covering x and not u. If c_1 1-covers y, then c_1 also 1-separates y and u; if c_1 does not 1-cover y, then c_1 also 1-separates y and x. In both cases, we are left with one pair of vertices not yet 1-separated by a codeword, which we can do with a second additional codeword c_2. Now $\mathcal{C} = \mathcal{C}_e \cup \{c_1, c_2\}$ is 1-identifying in G, and it has $|\mathcal{C}_e| + 2$ elements.

When at most two pairs of $(G, \mathcal{C}_e, 1)$-twins appear, then obviously with at most two more codewords added to $\mathcal{C}_e$ we can 1-separate them. $\triangle$

Note that we made no assumption on the connectivity of G_e. The following example shows that graphs G_e and G with $\gamma_1(G) = \gamma_1(G_e) + 2$ do exist; we do not know if this is the smallest possible example.

Example 19 *Let $G_e = (V, E_e)$ be the graph represented in Figure 6, and G the graph obtained by adding the edge $e = xy$. We claim that: (a) $\gamma_1(G_e) \leq 10$ and (b) $\gamma_1(G) \geq 12$, which by (15) implies that $\gamma_1(G) = 12 = \gamma_1(G_e) + 2$.*

Proof of (a). It is quite straightforward to check that $\mathcal{C}_e = \{1, 3, x, 6, 8, 8', 6', y, 3', 1'\}$ is 1-identifying in G_e. Hence $\gamma_1(G_e) \leq 10$.

Proof of (b). Let $\mathcal{C}$ be a 1-identifying code in G. Because 1 and 2 must be 1-separated by $\mathcal{C}$, we have $3 \in \mathcal{C}$; and because 1 must be 1-covered by at least one codeword, we have $1 \in \mathcal{C}$ or $2 \in \mathcal{C}$. Similarly, $\mathcal{C}$ contains $6, 6', 3'$ and at least one element in each of the 2-sets $\{7, 8\}$, $\{8', 7'\}$ and $\{2', 1'\}$, which amounts to eight codewords.

With simple arguments, we obtain the following fact:

- *there are at least three codewords in $\{1, 2, 3, 4, x\}$.*

The same is true for $\{x, 5, 6, 7, 8\}$, $\{y, 5', 6', 7', 8'\}$ and $\{y, 4', 3', 2', 1'\}$. So, if neither x nor y belongs to $\mathcal{C}$, there are at least $3 \times 4 = 12$ codewords, and we are done.

If, on the other hand, both x and y belong to $\mathcal{C}$, then we have already ten codewords, and still x, y and z are not 1-separated by any codeword; this will require two additional codewords, and again, $|\mathcal{C}| \geq 12$.

If we assume finally, without loss of generality, that $x \in \mathcal{C}$ and $y \notin \mathcal{C}$, then we have already chosen $(3 \times 2) + 5 = 11$ codewords: three in each of the *sets* $\{4', 3', 2', 1'\}$ *and* $\{5', 6', 7', 8'\}$*, one in each of the sets* $\{1, 2\}$ *and* $\{7, 8\}$*, plus* 3*,* 6 *and* x*; still,* x *and* z *are not* 1*-separated by any codeword, so again we need at least twelve codewords, which proves Claim (b).* $\triangle$

Next, we establish how large $\gamma_1(G_e) - \gamma_1(G)$ can be.

Theorem 20 *Let* $G = (V, E)$ *be a* 1*-twin-free graph with at least four vertices, let* x *and* y *be two vertices in* V *such that* $e = xy \in E$*, and let* $G_e = (V, E_e)$ *with* $E_e = E \setminus \{xy\}$*. Assume that* G_e *is also* 1*-twin-free.*

If $\mathcal{C}$ *is a* 1*-identifying code in* G*, then there exists a* 1*-identifying code* $\mathcal{C}_e$ *in* G_e *with*

$$|\mathcal{C}_e| \leq |\mathcal{C}| + 2.$$

As a consequence, we have:

$$\gamma_1(G_e) - \gamma_1(G) \leq 2. \tag{16}$$

Proof. We assume that $\mathcal{C}$ is not 1-identifying in G_e anymore, otherwise we are done. There can be two reasons why $\mathcal{C}$ is not 1-identifying:

1) at least one of the two vertices x and y, say x, is not 1-covered by any codeword anymore:

$$B_{G_e,1}(x) \cap \mathcal{C} = \emptyset = (B_{G,1}(x) \setminus \{y\}) \cap \mathcal{C},$$

which implies that $B_{G,1}(x) \cap \mathcal{C} = \{y\}$, $y \in \mathcal{C}$ and $x \notin \mathcal{C}$; we see that in this case y is still 1-covered by a codeword, namely itself.

If meanwhile all vertices remain 1-separated by $\mathcal{C}$ in G_e, then $\mathcal{C} \cup \{x\}$ is 1-identifying in G_e. But this first reason can go along with the second reason:

2) $(G_e, \mathcal{C}, 1)$-twins appear;

because only the edge xy is deleted when going from G to G_e, and similarly to the proof of Theorem 18, only the following pairs can be $(G_e, \mathcal{C}, 1)$-twins:

- x and y,
- x and u with $u \neq x, u \neq y$,
- y and v with $v \neq x, v \neq y$.

If x and y are $(G_e, \mathcal{C}, 1)$-twins, this means that

$$B_{G_e,1}(x) \cap \mathcal{C} = B_{G_e,1}(y) \cap \mathcal{C},$$

which implies that $x \notin \mathcal{C}$, $y \notin \mathcal{C}$, and so

$$B_{G,1}(x) \cap \mathcal{C} = B_{G,1}(y) \cap \mathcal{C},$$

contradicting the fact that $\mathcal{C}$ is 1-identifying in G.

Assume next that x and u are $(G_e, \mathcal{C}, 1)$-twins. Then

$$B_{G_e,1}(x) \cap \mathcal{C} = B_{G_e,1}(u) \cap \mathcal{C} = B_{G,1}(u) \cap \mathcal{C} \neq B_{G,1}(x) \cap \mathcal{C},$$

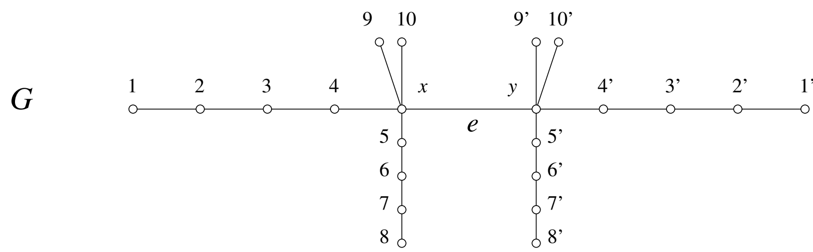

Figure 7: Graph G in Example 21.

and so

$$y \in \mathcal{C} \text{ and } B_{G_e,1}(x) \cap \mathcal{C} = (B_{G,1}(x) \cap \mathcal{C}) \setminus \{y\}.$$

If x and u are the only $(G_e, \mathcal{C}, 1)$-twins, then with two more codewords we can both 1-cover x if necessary and 1-separate x and u in G_e. The same argument would work if y and v were the only $(G_e, \mathcal{C}, 1)$-twins. So we assume that x and u, and y and v are $(G_e, \mathcal{C}, 1)$-twins. This implies that both x and y are codewords, each 1-covered by itself. All there is left to do is to 1-separate two pairs of $(G_e, \mathcal{C}, 1)$-twins in G_e, which can be done using two more codewords. $\triangle$

Note that we made no assumption on the connectivity of G and G_e. The following example shows that graphs G and G_e with $\gamma_1(G_e) = \gamma_1(G) + 2$ exist. Here the graph G_e is disconnected, but by adding a vertex which is linked to all the other vertices, we could also have, thanks to Lemmata 1 and 2, an example where G_e would be connected.

Example 21 *Let $G = (V, E)$ be the graph represented in Figure 7, and G_e the graph obtained by deleting the edge xy. We claim that: (a) $\gamma_1(G) \leq 12$ and (b) $\gamma_1(G_e) \geq 14$, which by (16) will imply that $\gamma_1(G_e) = 14 = \gamma_1(G) + 2$.*

Proof of (a). It is quite straightforward to check that $\mathcal{C} = \{1, 3, x, 6, 8, 9, 9', 8', 6', y, 3', 1'\}$ is 1-identifying in G. Hence $\gamma_1(G) \leq 12$.

Proof of (b). Let $\mathcal{C}_e$ be a 1-identifying code in G_e. We are going to show that the left part of the (now disconnected) graph G_e requires at least seven codewords.

As in Example 19, we have $3 \in \mathcal{C}_e$, $6 \in \mathcal{C}_e$, and $\mathcal{C}_e$ also contains at least one element in each of the 2-sets $\{1, 2\}$ and $\{7, 8\}$, which amounts to four codewords.

As in Example 19, we also have that:

- *there are at least three codewords in $\{1, 2, 3, 4, x\}$,*

and three codewords in $\{x, 5, 6, 7, 8\}$. So, if $x \notin \mathcal{C}_e$, there are, because of 9 and 10, at least $3 + 3 + 2 = 8$ codewords, and we are done. We now assume *that $x \in \mathcal{C}_e$, so that we have already taken five codewords. One more codeword is not sufficient to* 1*-separate both* 9 *and* 10*,* 9 *and x, and* 10 *and x. This proves Claim (b).* $\triangle$

By Theorems 18 and 20, we have the following result.

Theorem 22 *Let G_1 and G_2 be two* 1*-twin-free graphs, with same vertex set and differing by one edge. Then*

$$\gamma_1(G_1) - 2 \leq \gamma_1(G_2) \leq \gamma_1(G_1) + 2.$$

As a consequence, if for instance $\gamma_1(G_1) \leq a$ and $\gamma_1(G_2) \geq a + 2$, then $\gamma_1(G_1) = a$ and $\gamma_1(G_2) = a + 2$. $\triangle$

We conclude the case $r = 1$ by mentioning that pairs of graphs G and G_e such that $\gamma_1(G_e) - \gamma_1(G) = 0$ or $\gamma_1(G_e) - \gamma_1(G) = \pm 1$ exist.

Example 23 *We give simple examples with (a) $\gamma_1(G_e) - \gamma_1(G) = -1$ and (b) $\gamma_1(G_e) - \gamma_1(G) = 1$, omitting the easy case when the difference is* 0*.*

(a) Let $G_e = P_9 = x_1 x_2 \dots x_9$, and add the edge $\{x_3, x_5\}$ in order to obtain G. It is known ([3, Th. 3]) that $\gamma_1(P_9) = 5$, and it is easy to see that $\gamma_1(G) = 6$, so $\gamma_1(G_e) - \gamma_1(G) = -1$.

(b) Let G_e be the graph consisting of P_1 and P_4, and G be the graph obtained by adding an edge between one extremity of P_4 and the vertex of P_1, so that $G = P_5$. We have $\gamma_1(P_1) = 1$, $\gamma_1(P_4) = 3$, and $\gamma_1(P_5) = 3$, which shows that $\gamma_1(G_e) - \gamma_1(G) = 1$. $\triangle$

7 The case $r \geq 2$

We first give a result with $\gamma_r(G) - \gamma_r(G_e)$ equal to r; here, the graph G_e is connected, but we have found no better result using a disconnected graph.

Proposition 24 *There exist two (connected) r-twin-free graphs G and G_e such that*

$$\gamma_r(G) - \gamma_r(G_e) = r. \tag{17}$$

Proof. Let $r \geq 2$ and $p \geq 3$ be integers; for $1 \leq i \leq p$, we consider copies (V_i, E_i) of the path P_r, with the following notation:

$$V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,r}\}, \text{ and } E_i = \{\{v_{i,j}, v_{i,j+1}\} : 1 \leq j \leq r-1\}.$$

Let $G = (V, E)$, where

$$V = \cup_{1 \leq i \leq p} V_i \cup \{x\} \cup \{v\} \quad \text{and}$$

$$E = \cup_{1 \leq i \leq p} E_i \cup \{\{x, v_{i,1}\} : 1 \leq i \leq p\} \cup \{\{v, v_{i,r}\} : 1 \leq i \leq p\},$$

and build the graph G_e from G by deleting the edge $xv_{1,1}$, see Figure 8.

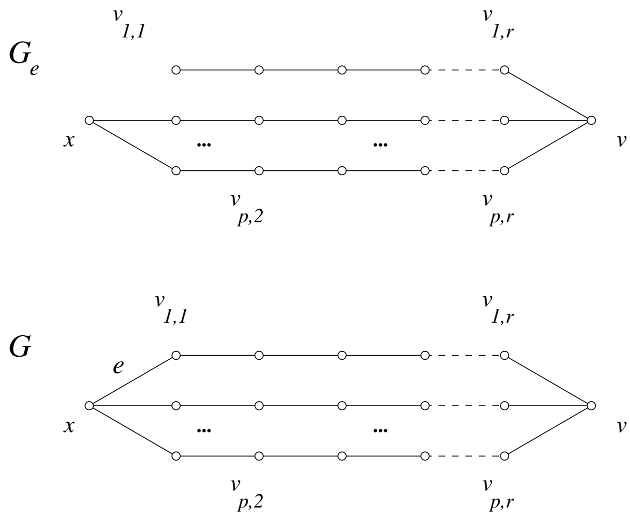

Figure 8: The graphs G_e and G in Proposition 24.

We claim that: (a) $\gamma_r(G) = r(p-1)+1$, and (b) $\gamma_r(G_e) = r(p-2)+1$, from which (17) follows.

Proof of (a). Note that in G, for i and j such that $1 \le i < j \le p$, the set of vertices

$$\{x\} \cup \{v_{i,k} : 1 \le k \le r\} \cup \{v\} \cup \{v_{j,k} : 1 \le k \le r\}$$

forms the cycle C_{2r+2}, which is r-twin-free and is denoted by $C(i,j)$. On such a cycle, we say that the vertex z is the *opposite* of the vertex y if z is the (only) vertex at distance $r+1$ from y.

We claim that, for k fixed between 1 and r, among the p vertices $v_{i,k}$, at least $p-1$ of them belong to any r-identifying code $\mathcal{C}$ in G. Indeed, assume on the contrary that two vertices, say $v_{1,k}$ and $v_{2,k}$, are not in $\mathcal{C}$; then their opposite vertices in $C(1,2)$, $v_{2,r-k+1}$ and $v_{1,r-k+1}$ respectively, cannot be r-separated by $\mathcal{C}$.

Finally, the fact that $B_{G,r}(v)\Delta B_{G,r}(x) = \{v,x\}$ (where Δ stands for the symmetric difference) shows that v or x belong to $\mathcal{C}$, and finally $\gamma_r(G) \ge (p-1)r+1$. On the other hand,

$$\{v\} \cup \{v_{i,k} : 2 \le i \le p, 1 \le k \le r\}$$

is an r-identifying code in G, with size $(p-1)r+1$, thus Claim (a) is proved. Note that this code contains all the vertices in G, except the $r+1$ vertices x and $v_{1,k}$, $1 \le k \le r$.

Proof of (b). The same argument about cycles of length $2r+2$ similarly shows that for k fixed between 1 and r, among the $p-1$ vertices $v_{i,k}$, $2 \le$

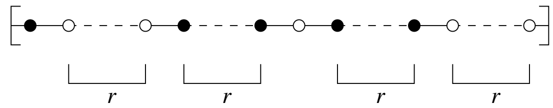

Figure 9: A pattern with $4r+2$ vertices, half of which are codewords.

$i \leq p$, at least $p-2$ of them belong to any r-identifying code in G_e. These $r(p-2)$ codewords are not sufficient, but adding $v_{1,1}$ gives an r-identifying code $\mathcal{C}_e$ in G_e, for example

$$\mathcal{C}_e = \{v_{1,1}\} \cup \{v_{i,k} : 3 \leq i \leq p, 1 \leq k \leq r\}.$$

△

Note that we could have contented ourselves with the inequalities $\gamma_r(G) \geq r(p-1)+1$ and $\gamma_r(G_e) \leq r(p-2)+1$, so as to obtain $\gamma_r(G) - \gamma_r(G_e) \geq r$.

We now investigate how large $\gamma_r(G_e) - \gamma_r(G)$ can be. We start with connected graphs.

7.1 Case of a connected graph G_e

Proposition 25 *There exist two (connected) r-twin-free graphs G and G_e such that*

$$\gamma_r(G_e) - \gamma_r(G) \geq r-1. \tag{18}$$

Proof. Consider the path P_{4r+2} represented in Figure 9, consisting of one codeword, then r noncodewords, then r codewords, then one noncodeword, then r codewords, then r noncodewords. This pattern was already described in [3, Fig. 2] and can be used to construct periodic r-identifying codes in chains or cycles, with density one half.

Now for $k \geq 2$, we let G_e be the cycle with $n = k(4r+2)+2r$ vertices and construct the graph G by adding the edge $e = x_n x_{4r+1}$, see Figure 10 for $k=2, r=4$ and $n=44$. We claim that the code $\mathcal{C}$ consisting of k concatenated patterns given by Figure 9, starting at x_r and ending at $x_{r+k(4r+2)-1}$, plus the codeword $x_{r+k(4r+2)}$, is r-identifying in G. Once we have proved this, and since $|\mathcal{C}| = k(2r+1)+1$ and $\gamma_r(G_e) = k(2r+1)+r$ by Theorem 3, we have proved (18).

Checking $\mathcal{C}$ can be done in two steps: first, it is easy to observe that the vertices x_i for $i \in \{r, \ldots, r+k(4r+2)-1\}$ are r-covered and pairwise r-separated by codewords, see also the proof of Th. 1 in [3]. Next, with the crucial help of the edge e, the $2r$ vertices $x_{k(4r+2)+r}$ to x_{r-1} are r-covered, pairwise r-separated, and r-separated fom the previous vertices by codewords. △

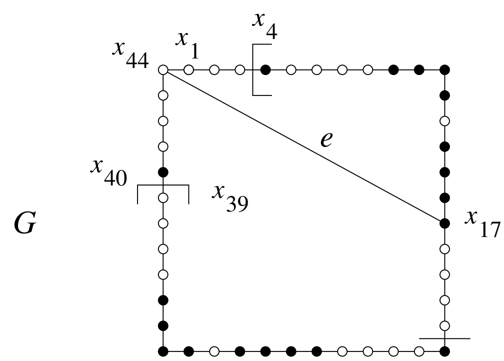

Figure 10: Graph G in Proposition 25, for $r = 4$ and $n = 44$. The 19 black vertices constitute a 4-identifying code in G.

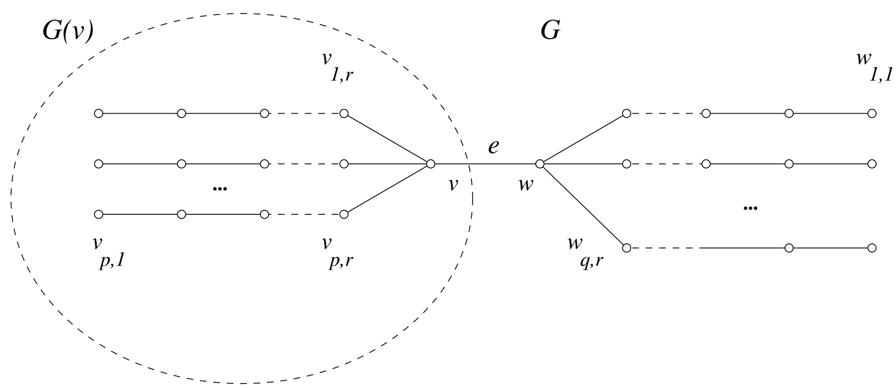

Figure 11: Graph G in Proposition 26.

7.2 Case of a disconnected graph G_e

We first give a result with $\gamma_r(G_e) - \gamma_r(G) \geq 2r - 3$, then investigate how to possibly reach $2r$, and finally give a conjecture.

Proposition 26 *There exist two r-twin-free graphs G and G_e such that*

$$\gamma_r(G_e) - \gamma_r(G) \geq 2r - 3. \tag{19}$$

Proof. Consider the graph G given by Figure 11, with the edge $e = vw$, where p and q are integers, $p \geq 3, q \geq 3$. We claim that: (a) $\gamma_r(G_e) \geq p + q + 4r - 6$, and (b) $\gamma_r(G) \leq p + q + 2r - 3$, from which (19) follows.

Proof of (a). If $G(v)$ represents the "left part" of G, it is sufficient to prove that $\gamma_r(G(v)) \geq p + 2r - 3$. The following two observations will be useful. For $1 \leq i \leq p$ and $2 \leq k \leq r$, we have:

$$B_{G(v),r}(v_{i,r-k+1}) \Delta B_{G(v),r}(v_{i,r-k+2}) = \{v_{j,k} : 1 \leq j \leq p, j \neq i\}, \tag{20}$$

and for $1 \le i < j \le p$:

$$B_{G(v),r}(v_{i,r}) \Delta B_{G(v),r}(v_{j,r}) = \{v_{i,1}, v_{j,1}\}. \tag{21}$$

The consequences are immediate. First, in order to have the vertices $v_{i,r}$, $1 \le i \le p$, pairwise r-separated in $G(v)$, we see by (21) that we need at least $p-1$ codewords among the p vertices $v_{i,1}$; second, for k fixed between 2 and r, we see, using (20), that we need at least two codewords among the p vertices $v_{i,k}$. So we obtain, after checking that $G(v)$ indeed is r-twin-free, that $\gamma_r(G(v)) \ge (p-1) + 2(r-1) = p + 2r - 3$, and Claim (a) is proved.

Proof of (b). We set $\mathcal{C} = \mathcal{C}_v \cup \mathcal{C}_w$, with

$$\mathcal{C}_v = \{v_{i,1} : 2 \le i \le p\} \cup \{v_{i,3}, v_{i,5}, \dots, v_{i,r} : 1 \le i \le 2\},$$

$$\mathcal{C}_w = \{w_{i,1} : 1 \le i \le q\} \cup \{w_{i,3}, w_{i,5}, \dots, w_{i,r} : 1 \le i \le 2\},$$

when r is odd, and when r is even:

$$\mathcal{C}_v = \{v_{i,1} : 2 \le i \le p\} \cup \{v_{i,3}, v_{i,5}, \dots, v_{i,r-1} : 1 \le i \le 2\} \cup \{v\},$$

$$\mathcal{C}_w = \{w_{i,1} : 1 \le i \le q\} \cup \{w_{i,3}, w_{i,5}, \dots, w_{i,r-1} : 1 \le i \le 2\} \cup \{w\}.$$

We show that $\mathcal{C}$, which has $p + q + 2r - 3$ elements, is r-identifying in G.

First, one vertex in the left part and one vertex in the right part of G cannot be $(G, \mathcal{C}, r)$-twins, thanks in particular to the q vertices $w_{i,1}$: more specifically, note that we take $w_{1,1}$ in $\mathcal{C}$ (among other candidates, such as $v_{1,1}$) in order to r-separate $w_{1,r}$ and $v_{1,r}$, and the consequence is that $\mathcal{C}_v$ and $\mathcal{C}_w$ are not completely alike. Second, two vertices in the left part of G cannot be $(G, \mathcal{C}, r)$-twins because

– on each of the p copies of P_r, consecutive vertices are r-separated alternatively by one codeword in $\mathcal{C}_w$ then one codeword in $\mathcal{C}_v$;

– two vertices on different copies j_1 and j_2 are r-separated by $v_{j_1,1}$ or $v_{j_2,1}$;

– and v is r-separated from the other vertices by $v_{2,1}$ or $v_{3,1}$.

The same argument works even better for the right part of G, by Lemma 5 and since $w_{1,1} \in \mathcal{C}_w$. And obviously, all vertices are r-covered by at least one codeword. △

We now give a lead on how (19) could be improved, in Example 30. To do this, we need some previously known results and one definition.

The *r-transitive closure*, or *r-th power*, of a graph $G = (V, E)$ is the graph denoted by $G^r = (V^r, E^r)$ and defined by $V^r = V$ and

$$E^r = \{xy : x \in V, y \in V, 0 < d_G(x, y) \le r\}.$$

The following very easy lemma can be found in [5] or [8].

Lemma 27 *In a graph $G = (V, E)$, a code $\mathcal{C} \subseteq V$ is r-identifying if and only if it is 1-identifying in G^r.* △

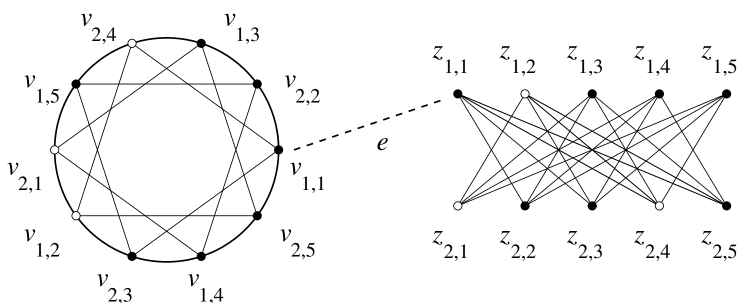

Figure 12: Two copies of G^*, a graph with 10 vertices and $\gamma_2(G^*) = 9$.

The following result is from [2], but we prefer to refer to the shorter, more elegant and more general proof of [14].

Theorem 28 *Any minimum* 1*-identifying code in a graph with* n *vertices has at most* $n-1$ *codewords.* $\triangle$

Corollary 29 *For all* $r \geq 1$*, any minimum* r*-identifying code in a graph with* n *vertices has at most* $n-1$ *codewords.* $\triangle$

Graphs G of order n with $\gamma_r(G) = n-1$ exist for all r and $n \geq 3r^2+1$, see Theorems 4–6 in [5] or [8], or Theorems 4 and 5 in [7]; on this topic, see also the more recent [13]. Our idea is to use two copies of such a "bad" graph, and to link them with an edge.

Example 30 *See Figure 12, where two different representations of the very same graph* G^* *with* 10 *vertices and* $\gamma_2(G^*) = 9$ *are given, forming a (disconnected) graph* G_e *with* 20 *vertices and* $\gamma_2(G_e) = 18$*. If we link any two vertices of the two copies, say* $v_{1,1}$ *and* $z_{1,1}$*, we obtain a graph* G *with* 20 *vertices and* $\gamma_2(G) \leq 14$*: a* 2*-identifying code in* G *is* $\mathcal{C} = \{v_{1,1}, v_{1,3}, v_{1,4}, v_{1,5}, v_{2,2}, v_{2,3}, v_{2,5}, z_{1,1}, z_{1,3}, z_{1,4}, z_{1,5}, z_{2,2}, z_{2,3}, z_{2,5}\}$*, see the black vertices in Figure 12. We have therefore an example with*

$$\gamma_2(G_e) - \gamma_2(G) \geq 4. \tag{22}$$

$\triangle$

We feel however that it is difficult to improve more than that (a decrease by 4 when $r = 2$, and, in the general case, by $2r$), and, together with the case $r = 1$, see (16) and Example 21 in Section 6, this suggests the following conjecture:

Conjecture 31 *For all* $r \geq 1$*, there exist two* r*-twin-free graphs* G_e *and* G *such that*

$$\gamma_r(G_e) - \gamma_r(G) = 2r. \tag{23}$$

Moreover, any two r-twin-free graphs G_e and G satisfy

$$\gamma_r(G_e) - \gamma_r(G) \leq 2r. \tag{24}$$

△

8 Conclusion

Table 2 recapitulates the results obtained in the previous sections, using (in)equalities (15)–(19) and (22), and also restates Conjecture 31. We only consider the difference $\gamma_r(G_e) - \gamma_r(G)$.

r	comment	$\gamma_r(G_e) - \gamma_r(G)$	reference
$= 1$	must be inside	$\{-2, -1, 0, 1, 2\}$	(15), (16), Th. 22
$= 1$	graphs with	$= -2, = -1, = 0, = 1, = 2$	Ex. 19, 21 and 23
≥ 2	(connected) graphs with	$\leq -r$	(17) in Prop. 24
≥ 2	connected graphs with	$\geq r - 1$	(18) in Prop. 25
≥ 2	graphs with	$\geq 2r - 3$	(19) in Prop. 26
$= 2$	graph with	≥ 4	(22) in Ex. 30
≥ 2	graphs with?	$= 2r$?	(23) in Conj. 31
≥ 2	impossible to have?	$> 2r$?	(24) in Conj. 31

Table 2: The difference $\gamma_r(G_e) - \gamma_r(G)$, as a function of r.

It would be interesting to try and see whether the results in Propositions 24, 25 and 26 could be improved, namely: for $r \geq 2$, are there graphs G and G_e with
(i) $\gamma_r(G_e) - \gamma_r(G) < -r$,
(ii) $\gamma_r(G_e) - \gamma_r(G) > r - 1$, or
(iii) $\gamma_r(G_e) - \gamma_r(G) > 2r - 3$,
where in Cases (i) and (iii), the graph G_e may be disconnected.

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