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Abstract

In many domains of information processing, bipolarity is a core feature to be considered: positive information represents what is possible or preferred, while negative information represents what is forbidden or surely false. If the information is moreover endowed with vagueness and imprecision, as is the case for instance in spatial information processing, then bipolar fuzzy sets constitute an appropriate knowledge representation framework. With the aim of extending the operations on such mathematical complex objects, we propose in this paper to extend mathematical morphology to bipolar fuzzy sets. This requires defining an appropriate lattice, on which algebraic dilations and erosions can be defined. We then address the case of operations based on a structuring element. These new operations have good properties, and can be useful in particular for processing spatial information, but also other types of bipolar information such as preferences and constraints.

Résumé

Dans beaucoup de domaines du traitement de l'information, la bipolarité est une caractéristique essentielle à prendre en compte : l'information positive représente ce qui est possible ou préféré, et l'information négative ce qui est interdit ou sûrement faux. De plus l'information peut être imprécise, comme c'est le cas par exemple en traitement d'informations spatiales, et les ensembles flous bipolaires constituent alors un modèle approprié de représentation des connaissances. Dans le but d'étendre des opérations à ces objets mathématiques complexes, nous proposons dans cet article d'étendre la morphologie mathématique aux ensembles flous bipolaires. Cela nécessite de définir un treillis approprié, sur lequel des dilatations er érosions algébriques peuvent être construites. Nous traitons ensuite le cas d'opérations définies à l'aide d'un élément structurant. Ces nouvelles opérations ont de bonnes propriétés, et sont utiles pour le traitement de l'information spatiale, mais aussi d'autres types d'informations comme des préférences et des contraintes.

Keywords: Bipolar information, bipolar fuzzy sets, mathematical morphology, complete lattice, dilation, erosion, spatial reasoning.
Mots clés : Information bipolaire, ensembles flous bipolaires, morphologie mathématique, treillis complets, dilatation, érosion, raisonnement spatial.

1 Introduction

A recent trend in contemporary information processing focuses on bipolar information, both from a knowledge representation point of view, and from a processing and reasoning one. Bipolarity is important to distinguish between (i) positive information, which represents what is guaranteed to be possible, for instance because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [55, 57]. This domain has recently motivated work in several directions, for instance for applications in knowledge representation, preference modeling, argumentation, multi-criteria decision analysis, cooperative games, among others [2, 10, 34, 39, 57, 60, 66, 68, 81, 82, 83]. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [9, 10, 56, 58], for asymmetric bipolar representations, which is the case addressed in this paper as well. Indeed, we consider the most general case where the positive and negative parts of information are not necessarily directly linked together, and may for instance come from completely different sources. Referring to the classification proposed in [58], the third class of bipolarity (asymmetric) should therefore be considered.

In this paper, we propose to handle such bipolar information using mathematical morphology operators. Mathematical morphology [88] has proved to be useful to process information in many different domains, such as image and vision [88, 89, 91], spatial reasoning [18, 28], preference modeling and logics (for fusion, revision, abduction, mediation...) [16, 29, 32, 33]. Extending mathematical morphology to bipolar information will therefore increase the modeling and reasoning capabilities in all these domains. This extension can be performed in a generic way, by defining a lattice as the underlying structure of bipolar knowledge representation (the interest of using complete lattices for mathematical morphology has been justified in [84]). The general framework of mathematical morphology leads to the definition of algebraic dilations and erosions, which are the two main operators, from which other ones can then be derived. This general formalism applies in different settings, and the proposed definitions can be specified for different types of lattices, e.g. based on bipolar sets, fuzzy sets or logical formulas. Bipolar fuzzy sets can be seen as a general structure covering several settings, and is therefore considered in this paper. Moreover, it allows handling an additional feature of imperfect information, related to its imprecision. Hence the proposed framework allows representing and dealing with both bipolarity and fuzziness.

Mathematical morphology on bipolar fuzzy sets was proposed for the first time in [19], by considering the complete lattice defined from the Pareto ordering. Then it was further developed, with additional properties, geometric aspects and applications to spatial reasoning, in [20, 21, 23, 24, 25]. The lexicographic ordering was considered too in [26]. Here we propose a more general algebraic setting and we show that the usual properties considered in mathematical morphology hold in any complete lattice representing bipolar information, whatever the choice of the partial ordering. This constitutes a substantial extension of our previous work, with more generality and more properties.

Recently, mathematical morphology on interval-valued fuzzy sets and intuitionistic fuzzy sets was addressed, independently, in [79], but without considering the algebraic framework of adjunctions, thus leading to weaker properties. This group then extended its approach with more properties in [74]. Pareto ordering was used in this work.

In Section 2, we set the algebraic framework, by defining a lattice structure on bipolar information, and introducing connectives. The remaining of the paper will rely on a representation of bipolar information as bipolar fuzzy sets, which encompasses several other models. We introduce definitions of algebraic dilations and erosions of bipolar fuzzy sets in Section 3, in a general way, whatever the chosen partial ordering. In the spatial domain, specific forms of these operators, involving a structuring element, are particularly interesting [88]. They are called morphological dilation and erosion. More generally they are useful in any application where some relation between elements of the underlying space should be involved. Morphological erosion and dilation are then defined in Section 4, and their properties are discussed. In the next two sections, we detail the case of two particular partial ordering: Pareto (or marginal) ordering in Section 5 and lexicographic ordering in Section 6. Finally, some derived operators are introduced in Section 7.

2 Algebraic framework

Mathematical morphology [88] usually relies on the algebraic framework of complete lattices, which has been justified in particular in [84], since it allows dealing properly with functions and bounded functions (which is particularly useful in the present context). It has also been extended to complete semi-lattices and general posets [67], based on the notion of adjunction [62] (see also [28] for a general description of the algebraic framework). In this paper, we only consider the case of complete lattices. We first introduce bipolar information models, and then a lattice structure on them, according to some partial ordering, which can be specified for any particular domain of application. Then bipolar connectives are defined.

2.1 Bipolar information

As mentioned in the introduction, bipolar information has two components, one related to positive information, and one related to negative information. These pieces of information can take different forms, according to the application domain, such as preferences and constraints, observations and rules, possible and forbidden places for an object in space, etc.

Let us assume that bipolar information is represented by a pair (μ, ν), where μ represents the positive information and ν the negative information, under a consistency constraint [55], which guarantees that the positive information is compatible with the constraints or rules expressed by the negative information. From a formal point of view, bipolar information can be represented in different mathematical frameworks, depending on the application domain, leading to different forms of μ and ν. Let us mention for instance:

- positive and negative information are subsets P and N of some set, and the consistency constraint is expressed as $P \cap N = \emptyset$, expressing that what is possible or preferred (positive information) should be included in what is not forbidden (negative information) [55];
- μ and ν are membership functions to fuzzy sets, defined over a space $\mathcal{S}$, and the consistency constraint is expressed as $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$ [19]. The pair (μ, ν) is then called a bipolar fuzzy set;
- positive and negative information are represented by logical formulas φ and ψ, generated by a set of propositional symbols and connectives, and the consistency constraint is then expressed as $\varphi \wedge \psi \models \perp$ (ψ represents what is forbidden or impossible);
- other examples include functions such as utility functions or capacities [60], preference functions [82], four-valued logics [68], possibility distributions [57, 58, 81].

In the following, we will detail the case of bipolar fuzzy sets, extending our previous work in [19, 20, 21, 23, 24, 25, 26] to any partial ordering. This case includes the other examples described above: the case of sets corresponds to the case where only bipolarity should be taken into account, without fuzziness (hence the membership function takes only values 0 and 1). In the case of logical formulas, we consider the models $[\![\varphi]\!]$ and $[\![\psi]\!]$ as sets or fuzzy sets. The lattice defined on the set of models is isomorphic to the one defined on $\Phi_{\equiv}$, where $\Phi_{\equiv}$ denotes the quotient space of set of formulas Φ by the syntactic equivalence relation between formulas (defined as $\varphi \equiv \varphi'$ iff $[\![\varphi]\!] = [\![\varphi']\!]$). Hence the case of bipolar fuzzy sets is general enough to cover several other mathematical settings.

Let $\mathcal{S}$ be the underlying space (the spatial domain for spatial information processing for instance).

Definition 1. *A bipolar fuzzy set on $\mathcal{S}$ is defined by an ordered pair of functions (μ, ν) from $\mathcal{S}$ into $[0, 1]$ such that $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$ (consistency constraint).*

Note that a bipolar fuzzy set is formally equivalent to an intuitionistic fuzzy set [6]. It is also equivalent to an interval-valued fuzzy set [99], where the interval at each point x is $[\mu(x), 1-\nu(x)]$ [54]. However the semantics are very different, and we keep here the terminology of bipolarity. A discussion on semantics is proposed in Section 2.4. An important point is that a bipolar fuzzy set is not one physical object (with potentially imprecisely defined membership function), but rather a complex mathematical object where μ and ν are really two different functions, which may represent different types of information or may be issued from different sources. The proposed approach also differs from the one in [100] where bipolarity is encoded on $[-1,0] \times [0,1]$ for defining bipolar fuzzy logic.

For each point x, $\mu(x)$ defines the membership degree of x (positive information) and $\nu(x)$ its non-membership degree (negative information). This formalism allows representing both bipolarity and fuzziness. Since the positive information models what is possible, preferred, observed or experienced, and the negative information what is forbidden or impossible, the consistency constraint avoids contradictions between what is forbidden and what is possible (i.e. the potential solutions should be included in what is not forbidden or impossible). The set of bipolar fuzzy sets defined on $\mathcal{S}$ is denoted by $\mathcal{B}$.

Let us denote by $\mathcal{L}$ the set of ordered pairs of numbers (a,b) in $[0,1]$ such that $a+b \leq 1$ (hence $(\mu,\nu) \in \mathcal{B} \Leftrightarrow \forall x \in \mathcal{S}, (\mu(x),\nu(x)) \in \mathcal{L}$). In all what follows, for each $(\mu,\nu) \in \mathcal{B}$, we will note $(\mu,\nu)(x) = (\mu(x),\nu(x))$ $(\in \mathcal{L})$, $\forall x \in \mathcal{S}$.

Note that fuzzy sets are particular cases of bipolar fuzzy sets, when $\forall x \in \mathcal{S}, \nu(x) = 1-\mu(x)$. Furthermore, if μ (and ν) only takes values 0 and 1, then bipolar fuzzy sets reduce to classical sets.

2.2 Partial ordering and lattice of bipolar fuzzy sets

Let $\preceq$ be a partial ordering on $\mathcal{L}$ such that $(\mathcal{L}, \preceq)$ is a complete lattice. We denote by $\bigvee$ and $\bigwedge$ the supremum and infimum, respectively. The smallest element is denoted by $0_{\mathcal{L}}$ and the largest element by $1_{\mathcal{L}}$. We denote by $\succeq$ the reverse order, i.e. $\forall ((a,b),(a',b')) \in \mathcal{L}^2, (a,b) \succeq (a',b') \Leftrightarrow (a',b') \preceq (a,b)$.

The partial ordering on $\mathcal{L}$ induces a partial ordering on $\mathcal{B}$, also denoted by $\preceq$ for sake of simplicity:

$$(\mu_1,\nu_1) \preceq (\mu_2,\nu_2) \text{ iff } \forall x \in \mathcal{S}, (\mu_1(x),\nu_1(x)) \preceq (\mu_2(x),\nu_2(x)). \tag{1}$$

Then $(\mathcal{B}, \preceq)$ is a complete lattice, for which the supremum and infimum are also denoted by $\bigvee$ and $\bigwedge$. The smallest element is the bipolar fuzzy set (μ_0,ν_0) taking value $0_{\mathcal{L}}$ at each point, and the largest element is the bipolar fuzzy set $(\mu_{\mathbb{I}},\nu_{\mathbb{I}})$ always equal to $1_{\mathcal{L}}$.

The following result is useful for proving the results in the next sections.

Lemma 1. *The following equivalence holds:*

$$\forall (a_1,b_1) \in \mathcal{L}, \forall (a_2,b_2) \in \mathcal{L},\ (a_1,b_1) \preceq (a_2,b_2) \Leftrightarrow \left\{ \begin{array}{l} (a_1,b_1) \vee (a_2,b_2) = (a_2,b_2) \\ (a_1,b_1) \wedge (a_2,b_2) = (a_1,b_1) \end{array} \right. \tag{2}$$

and similarly in $\mathcal{B}$:

$$\forall(\mu,\nu)\in\mathcal{B}, \forall(\mu',\nu')\in\mathcal{B},\ (\mu,\nu)\preceq(\mu',\nu')\Leftrightarrow\left\{\begin{array}{l}(\mu,\nu)\vee(\mu',\nu')=(\mu',\nu')\\(\mu,\nu)\wedge(\mu',\nu')=(\mu,\nu)\end{array}\right. \quad (3)$$

Proof. This result is directly derived from the fact that the supremum is the smallest upper bound and the infimum is the greatest lower bound. □

Note that the supremum and the infimum do not necessarily provide one of the input bipolar numbers or bipolar fuzzy sets (in particular if they are not comparable according to $\preceq$). However, they do if $\preceq$ is a total ordering.

2.3 Bipolar connectives

Let us now introduce some connectives, that will be useful in the following and that extend to the bipolar case the connectives classically used in fuzzy set theory [26]. In all what follows, increasingness and decreasingness are intended according to the partial ordering $\preceq$. Similar definitions can also be found e.g. in [44, 51] in the case of interval-valued fuzzy sets of intuitionistic fuzzy sets, for a specific partial ordering (Pareto-like ordering).

Definition 2. *A* **bipolar negation, or complementation***, on $\mathcal{L}$ is a decreasing operator N such that $N(0_\mathcal{L})=1_\mathcal{L}$ and $N(1_\mathcal{L})=0_\mathcal{L}$.*
In this paper, we restrict ourselves to involutive negations, such that $\forall(a,b)\in\mathcal{L}, N(N((a,b)))=(a,b)$ (these are the most interesting ones for mathematical morphology).

A **bipolar conjunction** *is an operator C from $\mathcal{L}\times\mathcal{L}$ into $\mathcal{L}$ such that $C(0_\mathcal{L},0_\mathcal{L})=C(0_\mathcal{L},1_\mathcal{L})=C(1_\mathcal{L},0_\mathcal{L})=0_\mathcal{L}$, $C(1_\mathcal{L},1_\mathcal{L})=1_\mathcal{L}$, and that is increasing in both arguments, i.e.: $\forall((a_1,b_1),(a_2,b_2),(a'_1,b'_1),(a'_2,b'_2))\in\mathcal{L}^4, (a_1,b_1)\preceq(a'_1,b'_1)$ and $(a_2,b_2)\preceq(a'_2,b'_2)\Rightarrow C((a_1,b_1),(a_2,b_2))\preceq C((a'_1,b'_1),(a'_2,b'_2))$.*

A **bipolar t-norm** *is a commutative and associative bipolar conjunction such that $\forall(a,b)\in\mathcal{L}, C((a,b),1_\mathcal{L})=C(1_\mathcal{L},(a,b))=(a,b)$ (i.e. the largest element of $\mathcal{L}$ is the unit element of C). If only the property on the unit element holds, then C is called a* **bipolar semi-norm***.*

A **bipolar disjunction** *is an operator D from $\mathcal{L}\times\mathcal{L}$ into $\mathcal{L}$ such that $D(1_\mathcal{L},1_\mathcal{L})=D(0_\mathcal{L},1_\mathcal{L})=D(1_\mathcal{L},0_\mathcal{L})=1_\mathcal{L}$, $D(0_\mathcal{L},0_\mathcal{L})=0_\mathcal{L}$, and that is increasing in both arguments.*

A **bipolar t-conorm** *is a commutative and associative bipolar disjunction such that $\forall(a,b)\in\mathcal{L}, D((a,b),0_\mathcal{L})=D(0_\mathcal{L},(a,b))=(a,b)$ (i.e. the smallest element of $\mathcal{L}$ is the unit element of D).*

A **bipolar implication** *is an operator I from $\mathcal{L}\times\mathcal{L}$ into $\mathcal{L}$ such that $I(0_\mathcal{L},0_\mathcal{L})=I(0_\mathcal{L},1_\mathcal{L})=I(1_\mathcal{L},1_\mathcal{L})=1_\mathcal{L}$, $I(1_\mathcal{L},0_\mathcal{L})=0_\mathcal{L}$ and that is decreasing in the first argument and increasing in the second argument.*

Proposition 1. *Bipolar connectives reduce to classical fuzzy connectives in the limit cases where there is no bipolarity in the input value and in the result.*

Let C be a bipolar t-norm. Then, under the non-bipolarity conditions, there exists a t-norm t such that $\forall(a_1, a_2) \in [0,1]^2$, $C((a_1, 1-a_1), (a_2, 1-a_2)) = (t(a_1, a_2), 1 - t(a_1, a_2))$. Similar expressions hold for the other connectives.

Proof. This result is directly derived from the definitions. $\square$

Proposition 2. *Any bipolar conjunction C has a null element, which is the smallest element of $\mathcal{L}$: $\forall(a,b) \in \mathcal{L}, C((a,b), 0_\mathcal{L}) = C(0_\mathcal{L}, (a,b)) = 0_\mathcal{L}$.*

Similarly, any bipolar disjunction has a null element, which is the largest element of $\mathcal{L}$: $\forall(a,b) \in \mathcal{L}, D((a,b), 1_\mathcal{L}) = D(1_\mathcal{L}, (a,b)) = 1_\mathcal{L}$.

For implications, we have $\forall(a,b) \in \mathcal{L}, I(0_\mathcal{L}, (a,b)) = I((a,b), 1_\mathcal{L}) = 1_\mathcal{L}$.

Proof. This follows directly from the monotony property and the boundary values of bipolar conjunctions and disjunctions: $\forall(a,b) \in \mathcal{L}, 0_\mathcal{L} \preceq (a,b) \preceq 1_\mathcal{L}$, and since any conjunction C is increasing, we have: $C(0_\mathcal{L}, 0_\mathcal{L}) \preceq C(0_\mathcal{L}, (a,b)) \preceq C(0_\mathcal{L}, 1_\mathcal{L})$. Since $C(0_\mathcal{L}, 0_\mathcal{L}) = C(0_\mathcal{L}, 1_\mathcal{L}) = 0_\mathcal{L}$, it follows that $C(0_\mathcal{L}, (a,b)) = 0_\mathcal{L}$. Similarly $C((a,b), 0_\mathcal{L}) = 0_\mathcal{L}$. The same reasoning applies for the proof for disjunctions and for implications. $\square$

As in the fuzzy case, conjunctions and implications may be related to each other based on the residuation principle, which corresponds to a notion of adjunction, which is also foundamental in mathematical morphology. This principle is expressed as follows in the bipolar case.

Definition 3. *A pair of bipolar connectives (I, C) forms an adjunction if: $\forall(a_i, b_i) \in \mathcal{L}, i = 1...3$,*

$$C((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2) \Leftrightarrow (a_3, b_3) \preceq I((a_1, b_1), (a_2, b_2)). \tag{4}$$

The connectives introduced in Definition 2 can be linked to each other in different ways (again this is similar to the fuzzy case).

Proposition 3. *The following properties hold:*

- *Given a bipolar t-norm C and a bipolar negation N, the following operator D defines a bipolar t-conorm: $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,*

$$D((a_1, b_1), (a_2, b_2)) = N(C(N((a_1, b_1)), N((a_2, b_2)))). \tag{5}$$

- *A bipolar implication I induces a bipolar negation N defined as:*

$$\forall(a,b) \in \mathcal{L}, N((a,b)) = I((a,b), 0_\mathcal{L}). \tag{6}$$

- *The following operator I_N, derived from a bipolar negation N and a bipolar conjunction C, defines a bipolar implication: $\forall((a_1, b_1), (a_2, b_2)) \in \mathcal{L}^2$,*

$$I_N((a_1, b_1), (a_2, b_2)) = N(C((a_1, b_1), N((a_2, b_2)))). \tag{7}$$

- *Conversely, a bipolar conjunction C can be defined from a bipolar negation N and a bipolar implication I:* $\forall((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,
$$C((a_1,b_1),(a_2,b_2)) = N(I((a_1,b_1),N((a_2,b_2)))). \tag{8}$$
- *Similarly, a bipolar implication can be defined from a negation N and a bipolar disjunction D as:* $\forall((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,
$$I_N((a_1,b_1),(a_2,b_2)) = D(N((a_1,b_1)),(a_2,b_2)). \tag{9}$$
- *A bipolar implication can also be defined by residuation from a bipolar conjunction C such that $\forall(a,b) \in \mathcal{L} \setminus 0_\mathcal{L}, C(1_\mathcal{L},(a,b)) \neq 0_\mathcal{L}$:* $\forall((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,
$$I_R((a_1,b_1),(a_2,b_2)) = \bigvee\{(a_3,b_3) \in \mathcal{L} \mid C((a_1,b_1),(a_3,b_3)) \preceq (a_2,b_2)\}. \tag{10}$$
The operators C and I_R are then said to be adjoint (see Definition 3).
- *Conversely, from a bipolar implication I_R such that $\forall(a,b) \in \mathcal{L} \setminus 1_\mathcal{L}, I_R(1_\mathcal{L},(a,b)) \neq 1_\mathcal{L}$, the conjunction C such that (C,I_R) forms an adjunction is given by:* $\forall((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,
$$C((a_1,b_1),(a_2,b_2)) = \bigwedge\{(a_3,b_3) \in \mathcal{L} \mid (a_2,b_2) \preceq I_R((a_1,b_1),(a_3,b_3))\}. \tag{11}$$

Proof. It is straightforward to show that the connectives defined by Equations 5, 6, 7, 8, and 9 satisfy all required properties, according to Definition 2.

Let us just detail the last two properties, involving the adjunction concept. Let C be a conjunction and I_R defined according to Equation 10. Then:
$$I_R(0_\mathcal{L},0_\mathcal{L}) = \bigvee\{(a_3,b_3) \in \mathcal{L} \mid C(0_\mathcal{L},(a_3,b_3)) \preceq 0_\mathcal{L}\}$$
and since $C(0_\mathcal{L},1_\mathcal{L}) = 0_\mathcal{L}$, the supremum is equal to $1_\mathcal{L}$.

Since $\forall((a,b),(a_3,b_3)) \in \mathcal{L}^2, C((a,b),(a_3,b_3)) \preceq 1_\mathcal{L}$, we have:
$$\forall(a,b) \in \mathcal{L}, I_R((a,b),1_\mathcal{L}) = \bigvee\{(a_3,b_3) \in \mathcal{L} \mid C((a,b),(a_3,b_3)) \preceq 1_\mathcal{L}\} = 1_\mathcal{L}.$$
For the last boundary condition, we have:
$$I_R(1_\mathcal{L},0_\mathcal{L}) = \bigvee\{(a_3,b_3) \in \mathcal{L} \mid C(1_\mathcal{L},(a_3,b_3)) \preceq 0_\mathcal{L}\}$$
and since $C(1_\mathcal{L},(a_3,b_3))$ cannot be equal to $0_\mathcal{L}$ except for $(a_3,b_3) = 0_\mathcal{L}$ by hypothesis, the supremum is equal to $0_\mathcal{L}$.

Finally the monotony properties directly result from the ones of C, and hence I_R is an implication.

Let us now show that the adjoint of C is actually I_R, as expressed in Equation 10. Let (I_R,C) be an adjoint pair. Then, $\forall(a_i,b_i) \in \mathcal{L}, i = 1...3$,
$$C((a_1,b_1),(a_3,b_3)) \preceq (a_2,b_2) \Leftrightarrow (a_3,b_3) \preceq I((a_1,b_1),(a_2,b_2)).$$

Hence

$$\bigvee\{(a_3,b_3)\in\mathcal{L}\mid C((a_1,b_1),(a_3,b_3))\preceq(a_2,b_2)\}\preceq I_R((a_1,b_1),(a_2,b_2)).$$

Conversely, from the tautology $I_R((a_1,b_1),(a_2,b_2))\preceq I_R((a_1,b_1),(a_2,b_2))$ we derive, by applying the adjunction property and setting $I_R((a_1,b_1),(a_2,b_2))=(a_3,b_3)$:

$$C((a_1,b_1),I_R((a_1,b_1),(a_2,b_2)))\preceq(a_2,b_2)\Rightarrow$$

$$I_R((a_1,b_1),(a_2,b_2))\preceq\bigvee\{(a_3,b_3)\mid C((a_1,b_1),(a_3,b_3)))\preceq(a_2,b_2)\}.$$

Hence $I_R=\bigvee\{(a_3,b_3)\in\mathcal{L}\mid C((a_1,b_1),(a_3,b_3))\preceq(a_2,b_2)\}$.

The proof for the last property is similar. Note that the condition on I_R makes $C(1_\mathcal{L},1_\mathcal{L})=1_\mathcal{L}$ hold.

□

Proposition 4. *Let C be a bipolar conjunction and I a bipolar implication derived from C, either as I_N using an involutive negation (Equation 7) or as I_R by residuation (Equation 10). The following equivalence holds:*

$$\forall(a,b)\in\mathcal{L},C(1_\mathcal{L},(a,b))=(a,b)\Leftrightarrow\forall(a,b)\in\mathcal{L},I(1_\mathcal{L},(a,b))=(a,b),\qquad(12)$$

i.e. C admits $1_\mathcal{L}$ as unit element on the left iff I admits $1_\mathcal{L}$ as unit element on the left.

Proof. This result directly follows from Equations 7, 8, 10, and 11. □

Proposition 5. *If C and I are bipolar connectives such that (I,C) forms an adjunction (i.e. verifies Equation 4), then C distributes over the supremum and I over the infimum on the right, i.e.:*

$$\forall(a_i,b_i)\in\mathcal{L},\forall(a,b)\in\mathcal{L},$$

$$\bigvee_i C((a,b),(a_i,b_i))=C((a,b),\bigvee_i(a_i,b_i)),\qquad(13)$$

$$\bigwedge_i I((a,b),(a_i,b_i))=I((a,b),\bigwedge_i(a_i,b_i)).\qquad(14)$$

Proof. The proof is similar to the classical proof done in mathematical morphology to show that an adjunction defines a dilation (i.e. an operation that commutes with the supremum) and an erosion (see Section 3 for these definitions) [61]. Let us assume that (I,C) is an adjunction. The following equivalences hold, for any $(a,b),(\alpha,\beta)$ and any family (a_i,b_i) in $\mathcal{L}$:

$$\begin{aligned}
(1)\ \bigvee_i C((a,b),(a_i,b_i)) \preceq (\alpha,\beta) &\Leftrightarrow \forall i, C((a,b),(a_i,b_i)) \preceq (\alpha,\beta) \\
\text{(adjunction property)} &\Leftrightarrow \forall i, (a_i,b_i) \preceq I((a,b),(\alpha,\beta)) \\
&\Leftrightarrow \bigvee_i (a_i,b_i) \preceq I((a,b),(\alpha,\beta)) \\
\text{(adjunction property)} &\Leftrightarrow C((a,b),\bigvee_i (a_i,b_i)) \preceq (\alpha,\beta)\ (2)
\end{aligned}$$

Let $(\alpha,\beta) = \bigvee_i C((a,b),(a_i,b_i))$. Then (1) trivially holds, and therefore (2) holds too, i.e. $C((a,b),\bigvee_i(a_i,b_i)) \preceq \bigvee_i C((a,b),(a_i,b_i))$. Now let $(\alpha,\beta) = C((a,b),\bigvee_i(a_i,b_i))$. Then (2) holds, hence (1) holds, i.e. $\bigvee_i C((a,b),(a_i,b_i)) \preceq C((a,b),\bigvee_i(a_i,b_i))$. Finally we have $\bigvee_i C((a,b),(a_i,b_i)) = C((a,b),\bigvee_i(a_i,b_i))$, i.e. C commutes with the supremum on the right.

The proof for I is similar. □

Note that the distributivity on the left requires C to be commutative, and in that case we also have:

$$\bigvee_i C((a_i,b_i),(a,b)) = C(\bigvee_i (a_i,b_i),(a,b)), \tag{15}$$

and then we have in a similar way for I:

$$\bigwedge_i I((a_i,b_i),(a,b)) = I(\bigvee_i (a_i,b_i),(a,b)). \tag{16}$$

The following properties of adjunctions will also be useful for deriving mathematical morphology operators.

Proposition 6. *Let (I,C) be an adjunction. Then the following properties hold:*

- *C is increasing in the second argument and I in the second one. If furthermore C is commutative, then it is also increasing in the first one.*
- *$0_{\mathcal{L}}$ is the null element of C on the right and $1_{\mathcal{L}}$ is the null element of I on the right, i.e.*

$$\forall (a,b) \in \mathcal{L}, C((a,b),0_{\mathcal{L}}) = 0_{\mathcal{L}},\ I((a,b),1_{\mathcal{L}}) = 1_{\mathcal{L}}.$$

Proof. From Lemma 1, and from the distributivity of C over the supremum on the right if (I,C) is an adjunction (Proposition 5), it follows that:

$$\forall ((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2,\ (a_1,b_1) \preceq (a_2,b_2) \Rightarrow$$

$$\forall (a,b) \in \mathcal{L},\ C((a,b),(a_1,b_1)) \vee C((a,b),(a_2,b_2)) = C((a,b),(a_1,b_1) \vee (a_2,b_2)) = C((a,b),(a_2,b_2))$$

hence $C((a,b),(a_1,b_1)) \preceq C((a,b),(a_2,b_2))$. Obviously if C is commutative, the increasingness with respect to the first argument follows directly.

The proof for I is similar.

From the adjunction property and the fact that $0_{\mathcal{L}}$ is the smallest element, we can write:

$$\forall ((a,b),(a',b')) \in \mathcal{L}^2,\ 0_{\mathcal{L}} \preceq I((a,b),(a',b')) \Rightarrow C((a,b),0_{\mathcal{L}}) \preceq (a',b')$$

hence $C((a,b),0_{\mathcal{L}})$ has to be the smallest element, i.e. $0_{\mathcal{L}}$.

Similarly for I, we show that $\forall (a,b) \in \mathcal{L},\ I((a,b),1_{\mathcal{L}})$ has to be the largest elements, i.e. $1_{\mathcal{L}}$. □

Finally, some ordering properties hold with respect to the infimum and the supremum of the lattice $(\mathcal{L},\preceq)$. More ordering properties can be exhibited for specific orderings, as we will see later on for the Pareto ordering.

Proposition 7.

- *Let C be a bipolar conjunction that admits $1_{\mathcal{L}}$ as unit element. Then*

$$\forall ((a,b),(a',b')) \in \mathcal{L}^2,\ C((a,b),(a',b')) \preceq (a,b) \wedge (a',b').$$

- *Let I be a bipolar implication that admits $1_{\mathcal{L}}$ as unit element on the left. Then*

$$\forall ((a,b),(a',b')) \in \mathcal{L}^2,\ (a',b') \preceq I((a,b),(a',b')).$$

- *Let I be a bipolar implication that admits $0_{\mathcal{L}}$ as unit element on the right. Then*

$$\forall ((a,b),(a',b')) \in \mathcal{L}^2,\ (a,b) \preceq I((a,b),(a',b')).$$

Proof. Let C be a bipolar conjunction that admits $1_{\mathcal{L}}$ as unit element. Since C is increasing, we have:

$$\forall ((a,b),(a',b')) \in \mathcal{L}^2,\ C((a,b),(a',b')) \preceq C((a,b),1_{\mathcal{L}})$$

and $C((a,b),1_{\mathcal{L}}) = (a,b)$ under the hypothesis. Similarly, $C((a,b),(a',b')) \preceq (a',b')$ and the first result follows.

The two results on I are derived in a similar way, by using the decreasingness of I with respect to the first argument and its increasingness with respect to the second one. □

2.4 A few comments about semantics

It is interesting to note that bipolar fuzzy sets are formally linked to intuitionistic fuzzy sets [6], interval-valued fuzzy sets [99] and vague sets, or to clouds when boundary constraints are added [52, 80], as shown by several authors [38, 54]. However, their respective semantics differ. Concerning intuitionistic fuzzy sets, there has been a lot of discussions about terminology in this domain recently [7, 54], mainly because of the word "intuitionistic" which is misleading and introduces confusions with intuitionistic logics. The semantics of intuitionistic fuzzy sets reveals the possibility of some indetermination between membership and non-membership to a set. As for interval-valued fuzzy sets or clouds, their semantics correspond to the representation of some imprecision or uncertainty about the membership value, which can only be given as an interval, and not as a crisp number.

However the semantics of bipolar fuzzy sets is different. A bipolar fuzzy set in the spatial domain does not necessarily represent one physical object or spatial entity, but rather more complex information, potentially issued from different sources. This refers to the third type of bipolarity, according to the classification presented in [55, 56]. What we call a bipolar fuzzy set is then a mathematical complex object, not a physical object. An example is the modeling of information concerning the location of a robot: positive information could concern potential locations (for instance derived from sensor data) and negative information could concern forbidden places (because they are already occupied by other objects, or the robot is not allowed to move there, etc.). Let us consider another example, in a different domain, about preference modeling [9]. Positive information describes what is desired and allows sorting solutions, while negative information describes what is rejected or unacceptable and defines constraints to be fulfilled. The gap between positive and negative information does not necessarily concern indetermination, but rather neutrality or indifference.

2.5 A note on partial ordering

One of the main issues in the proposed extensions of mathematical morphology to bipolar information is to handle the two components (i.e. positive and negative information) and to define an adequate and relevant ordering. Two extreme cases are Pareto ordering (also called marginal ordering) and lexicographic ordering. The Pareto ordering handles both components in a symmetric way, while the lexicographic ordering on the contrary gives a strong priority to one component, and the other one is then seldom considered. These features can be seen as either advantages or drawbacks, depending on the context and on the application. This issue has been addressed in other types of work, where different partial orderings have been discussed. We mention here two examples: color image processing and social choice.

Color image processing: The question of defining a suitable ordering on vectorial images (in particular color images) has been widely addressed in the mathematical imaging community (see e.g. [4] for a review). The lexicographic ordering is known to excessively privilege one of the colors, and can be refined for instance by defining a rougher quantization on the first color, so as to have more frequent comparisons based on the two other ones. Other approaches use a 1D scale by applying a scalar function to the vectors, define groups of vectors which are then ranked, or base the comparison on a distance to a reference vector, just to cite a few ones [3, 4, 5].

Social choice: This is another domain where the question of defining a partial ordering is crucial, in particular for multi-criteria decision making or voting problems. Various orderings have been proposed, including refinements of the lexicographic ordering, leximin/leximax, discrimax, tolerant Pareto, etc. [37, 53, 59, 77, 86].

All these works can guide the choice of an ordering adapted to bipolar information.

The two following sections remain general, and apply to any partial ordering, while two specific examples will be detailed next: Pareto ordering in Section 5 and lexicographic ordering in Section 6.

3 Algebraic dilations and erosions of bipolar fuzzy sets

Once we have a complete lattice, it is easy to define algebraic dilations and erosions on this lattice, as classically done in mathematical morphology [61, 62, 89]. Here we only consider operations from the lattice $(\mathcal{B}, \preceq)$ into itself.

Definition 4. *A dilation is an operator δ from $\mathcal{B}$ into $\mathcal{B}$ that commutes with the supremum:*

$$\forall (\mu_i, \nu_i) \in \mathcal{B},\ \delta(\bigvee_i (\mu_i, \nu_i)) = \bigvee_i \delta((\mu_i, \nu_i)), \tag{17}$$

where (μ_i, ν_i) is any family (finite or not) of elements of $\mathcal{B}$.
An erosion is an operator ε from $\mathcal{B}$ into $\mathcal{B}$ that commutes with the infimum:

$$\forall (\mu_i, \nu_i) \in \mathcal{B},\ \varepsilon(\bigwedge_i (\mu_i, \nu_i)) = \bigwedge_i \varepsilon((\mu_i, \nu_i)). \tag{18}$$

The following results are directly derived from the properties of complete lattices [61, 62] (they hold for any dilation and erosion defined on any complete lattice, hence on $\mathcal{B}$).

Proposition 8. *Algebraic dilations δ and erosions ε on $\mathcal{B}$ satisfy the following properties:*

- *δ and ε are increasing operators;*
- *δ preserves the smallest element: $\delta((\mu_0, \nu_0)) = (\mu_0, \nu_0)$;*
- *ε preserves the largest element: $\varepsilon((\mu_{\mathbb{I}}, \nu_{\mathbb{I}})) = (\mu_{\mathbb{I}}, \nu_{\mathbb{I}})$;*
- *by denoting (μ_x, ν_x) the canonical bipolar fuzzy set associated with (μ, ν) and x such that $(\mu_x, \nu_x)(x) = (\mu(x), \nu(x))$ and $\forall y \in \mathcal{S} \setminus \{x\}, (\mu_x, \nu_x)(y) = 0_{\mathcal{L}}$, we have $(\mu, \nu) = \bigvee_x (\mu_x, \nu_x)$ and $\delta((\mu, \nu)) = \bigvee_x \delta((\mu_x, \nu_x))$.*

The last result leads to morphological operators in case $\delta((\mu_x, \nu_x))$ has the same "shape" everywhere (and is then a bipolar fuzzy structuring element). This case is detailed in Section 4.

A fundamental notion in this algebraic framework is the one of adjunction.

Definition 5. *A pair of operators (ε, δ) defines an adjunction on $(\mathcal{B}, \preceq)$ iff:*

$$\forall (\mu, \nu) \in \mathcal{B}, \forall (\mu', \nu') \in \mathcal{B},\ \delta((\mu, \nu)) \preceq (\mu', \nu') \Leftrightarrow (\mu, \nu) \preceq \varepsilon((\mu', \nu')) \tag{19}$$

Again we can derive a series of results from the properties of complete lattices and adjunctions [61, 62].

Proposition 9. *If a pair of operators (ε, δ) on $\mathcal{B}$ defines an adjunction, then the following results hold:*

- *δ preserves the smallest element and ε the largest element of the lattice;*
- *δ is a dilation and ε is an erosion, in the sense of Definition 4;*
- *$\delta\varepsilon$ is anti-extensive: $\delta\varepsilon \preceq Id$, where Id denotes the identity mapping on $\mathcal{B}$ (i.e. $\forall (\mu, \nu) \in \mathcal{B}, Id(\mu, \nu) = (\mu, \nu)$), and $\varepsilon\delta$ is extensive: $Id \preceq \varepsilon\delta$ (the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are called morphological opening and morphological closing, respectively);*
- *$\delta\varepsilon\delta\varepsilon = \delta\varepsilon$ and $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$, i.e. morphological opening and closing are idempotent operators.*

Proposition 10. *Let δ and ε be two increasing operators such that $\delta\varepsilon$ is anti-extensive and $\varepsilon\delta$ is extensive. Then (ε, δ) is an adjunction.*

The following representation result also holds.

Proposition 11. *If ε is an increasing operator, it is an algebraic erosion if and only if there exists δ such that (ε, δ) is an adjunction. The operator δ is then an algebraic dilation and can be expressed as:*

$$\delta((\mu, \nu)) = \bigwedge \{(\mu', \nu') \in \mathcal{B} :\ (\mu, \nu) \preceq \varepsilon((\mu', \nu'))\}. \tag{20}$$

A similar representation result holds for erosion.

Proofs are omitted in this section, since they are exactly the same as in any complete lattice, and there is nothing specific to do for the particular case of the lattice $(\mathcal{B}, \preceq)$.

4 Morphological dilations and erosions of bipolar fuzzy sets

Particular forms of dilations and erosions, called morphological dilations and erosions, are defined in classical morphology, involving the notion of structuring element [88].

In the spatial domain $\mathcal{S}$ for instance ($\mathcal{S}$ is then assumed to be an affine space or at least a space where translations can be defined), a structuring element is a subset of $\mathcal{S}$ with fixed shape and size, directly influencing the spatial extent of the morphological transformations. It is generally assumed to be compact, so as to guarantee good properties. In the discrete case, it is often assumed to be connected, in the sense of a discrete connectivity defined on $\mathcal{S}$. The general principle underlying morphological operators, under an assumption of invariance by translation, consists in translating the structuring element at every position in space and checking if this translated structuring element satisfies some relation with the original set (inclusion for erosion, intersection for dilation) [61, 62, 85, 88, 89]. This principle has also been used in the main extensions of mathematical morphology to fuzzy sets [30, 49, 50, 72, 78, 90]. It has been further investigated in the algebraic framework of quantales [1, 87, 92].

More generally, without any assumption on the underlying domain $\mathcal{S}$, a structuring element is defined as a binary relation between two elements of $\mathcal{S}$ (i.e. y is in relation with x if and only if $y \in B_x$) [28]. This allows on the one hand dealing with spatially varying structuring elements (when $\mathcal{S}$ is the spatial domain), as e.g. in [11, 35, 36, 42, 48, 69], or with graph structures (e.g. [46, 76, 94, 96]), and on the other hand establishing interesting links with several other domains, such as rough sets [15], formal logics [16, 29, 31, 33], and, in the more general case where the morphological operations are defined from one set to another one, with Galois connections and formal concept analysis, as shown e.g. in [28].

From now on, we assume that $\mathcal{S}$ is an affine space on which translations are defined (but all definitions and results also apply to the other situations mentioned above). Following the same principle as in classical morphology, defining morphological erosions of bipolar fuzzy sets, using bipolar fuzzy structuring elements, requires to define a degree of inclusion between bipolar fuzzy sets. Such inclusion degrees have been proposed in the context of intuitionistic fuzzy sets and interval-valued fuzzy sets [44, 51]. With our notations, a degree of inclusion of a bipolar fuzzy set (μ', ν') in another bipolar fuzzy set (μ, ν) is defined as [19]:

$$\bigwedge_{x \in \mathcal{S}} I((\mu'(x), \nu'(x)), (\mu(x), \nu(x))) \tag{21}$$

where I is a bipolar implication, and a degree of intersection is defined as:

$$\bigvee_{x \in \mathcal{S}} C((\mu'(x), \nu'(x)), (\mu(x), \nu(x))) \tag{22}$$

where C is a bipolar conjunction. Note that both inclusion and intersection degrees are elements of $\mathcal{L}$, i.e. they are defined as bipolar degrees.

Based on these concepts, we can now propose a general definition for morphological erosions and dilations, thus extending our previous work in [19, 20, 21, 26].

Definition 6. *Let (μ_B, ν_B) be a bipolar fuzzy structuring element (in $\mathcal{B}$). The erosion of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from a bipolar implication I as:*

$$\forall x \in \mathcal{S},\ \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \bigwedge_{y \in \mathcal{S}} I((\mu_B(y-x), \nu_B(y-x)), (\mu(y), \nu(y))). \quad (23)$$

In this equation, $\mu_B(y-x)$ (respectively $\nu_B(y-x)$) represents the value at point y of the translation of μ_B (respectively ν_B) at point x.

Definition 7. *Let (μ_B, ν_B) be a bipolar fuzzy structuring element (in $\mathcal{B}$). The dilation of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from a bipolar conjunction C as:*

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \bigvee_{y \in \mathcal{S}} C((\mu_B(x-y), \nu_B(x-y)), (\mu(y), \nu(y))). \quad (24)$$

Proposition 12. *Definitions 6 and 7 are consistent: they actually provide bipolar fuzzy sets of $\mathcal{B}$, i.e. $\forall (\mu,\nu) \in \mathcal{B}, \forall (\mu_B,\nu_B) \in \mathcal{B}, \delta_{(\mu_B,\nu_B)}((\mu,\nu)) \in \mathcal{B}$ and $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \in \mathcal{B}$.*

Proof. The degrees of inclusion and intersection are elements of $\mathcal{L}$. Therefore $\forall x \in \mathcal{S}, \delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) \in \mathcal{L}$ and $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x)$, i.e. $\delta_{(\mu_B,\nu_B)}((\mu,\nu)) \in \mathcal{B}$ and $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \in \mathcal{B}$. □

Proposition 13. *In case the bipolar fuzzy sets are usual fuzzy sets (i.e. $\nu = 1-\mu$ and $\nu_B = 1-\mu_B$), the definitions lead to the usual definitions of fuzzy dilations and erosions. Hence they are also compatible with classical morphology in case μ and μ_B are crisp.*

Proof. This directly follows from the boundary constraints in the definitions of bipolar conjunctions and implications. □

Proposition 14. *Definitions 6 and 7 provide an adjunction (ε, δ) if and only if (I, C) is an adjunction.*

Proof. Let us first assume that (I, C) is an adjunction. Expressing Equation 4 for $(a_1, b_1) = (\mu_B, \nu_B)(x-y), (a_3, b_3) = (\mu, \nu)(y), (a_2, b_2) = (\mu', \nu')(x)$ we obtain, for all $(\mu, \nu), (\mu', \nu'), (\mu_B, \nu_B)$ in $\mathcal{B}$, and all x, y in $\mathcal{S}$:

$$C((\mu_B,\nu_B)(x-y), (\mu,\nu)(y)) \preceq (\mu',\nu')(x) \Leftrightarrow (\mu,\nu)(y) \preceq I((\mu_B,\nu_B)(x-y), (\mu',\nu')(x))$$

and, by taking the supremum over y on the left hand side of the equivalence and the infimum over x on the right hand side, we get:

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu)) \preceq (\mu',\nu') \Leftrightarrow (\mu,\nu) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu',\nu'))$$

hence (ε, δ) is an adjunction.

Conversely, let us assume that (ε, δ) is an adjunction. Expressing the adjunction equivalence for bipolar fuzzy sets taking constant values ($\forall x \in \mathcal{S}, (\mu_B, \nu_B)(x) = (a_1, b_1), (\mu, \nu)(x) = (a_2, b_2), (\mu', \nu')(x) = (a_3, b_3)$), we derive immediately the adjunction equivalence for I and C. $\square$

Proposition 15. *If I and C are bipolar connectives such that (I, C) is an adjunction, then the operator ε defined from I by Equation 23 commutes with the infimum and the operator δ defined from C by Equation 24 commutes with the supremum, i.e. they are algebraic erosion and dilation. Moreover they are increasing with respect to (μ, ν).*

Proof. Let I and C be bipolar connectives forming an adjunction. From Proposition 5, it follows that I (respectively C) commutes with the infimum (respectively with the supremum) on the right. Applying these properties to the expressions of ε and δ as given in Equations 23 and 24 leads to the result. The increasingness property directly follows, as in any lattice (see Proposition 8). $\square$

Proposition 16. *If (I, C) is an adjunction such that C is increasing in the first argument and I is decreasing in the first argument (typically if they are a bipolar conjunction and a bipolar implication), then the operator ε defined from I by Equation 23 is decreasing with respect to the bipolar fuzzy structuring element and the operator δ defined from C by Equation 24 is increasing with respect to the bipolar fuzzy structuring element.*

Proof. These monotony properties are directly derived from the ones of C, I, $\vee$ and $\wedge$. $\square$

Proposition 17. *C distributes over the supremum and I over the infimum on the right if and only if ε and δ defined by Equations 23 and 24 are algebraic erosion and dilation, respectively.*

Proof. Let us assume that C and I distribute on the right over the supremum and the infimum respectively. Then for all (μ_B, ν_B) in $\mathcal{B}$ and for any family (μ_i, ν_i) in $\mathcal{B}$, the following equalities hold, $\forall x \in \mathcal{S}$:

$$\begin{aligned}
\delta_{(\mu_B,\nu_B)}(\vee_i(\mu_i,\nu_i))(x) &= \vee_{y\in\mathcal{S}} C((\mu_B,\nu_B)(x-y), \vee_i(\mu_i,\nu_i)(y)) \\
&= \vee_{y\in\mathcal{S}} \vee_i C((\mu_B,\nu_B)(x-y), (\mu_i,\nu_i)(y)) \\
&= \vee_i \delta_{(\mu_B,\nu_B)}((\mu_i,\nu_i))(x) \\
\varepsilon_{(\mu_B,\nu_B)}(\wedge_i(\mu_i,\nu_i))(x) &= \wedge_{y\in\mathcal{S}} I((\mu_B,\nu_B)(y-x), \wedge_i(\mu_i,\nu_i)(y)) \\
&= \wedge_{y\in\mathcal{S}} \wedge_i I((\mu_B,\nu_B)(y-x), (\mu_i,\nu_i)(y)) \\
&= \wedge_i \varepsilon_{(\mu_B,\nu_B)}((\mu_i,\nu_i))(x)
\end{aligned}$$

Hence the distributivity of C and I entails the commutativity of δ with the supremum and the one of ε with the infimum, i.e. δ is a dilation and ε is an erosion.

Conversely, if δ is an algebraic dilation and ε is an algebraic erosion (i.e. they commute with the supremum and the infimum, respectively), then by applying this property to bipolar fuzzy sets taking constant values, the distributivity of C over the supremum on the right and the distributivity of I over the infimum on the right directly follow. $\square$

In the following, we only consider cases where the definitions actually provide algebraic dilations and erosions (which are the only ones that are interesting). Obviously, all results of Section 3 also hold.

Note that while δ commutes with the supremum and ε with the infimum, the converse is generally not true. However, inequalities hold, as in classical morphology.

Proposition 18. *Let δ and ε be a dilation and an erosion defined by Equations 24 and 23. Then, for all $(\mu_B, \nu_B), (\mu, \nu), (\mu', \nu')$ in $\mathcal{B}$, we have:*

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu) \wedge (\mu',\nu')) \preceq \delta_{(\mu_B,\nu_B)}((\mu,\nu)) \wedge \delta_{(\mu_B,\nu_B)}((\mu',\nu')), \tag{25}$$

$$\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \vee \varepsilon_{(\mu_B,\nu_B)}((\mu',\nu')) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu) \vee (\mu',\nu')). \tag{26}$$

Proof. The results are derived from the increasingness of C and the increasingness of I with respect to the second argument. $\square$

Proposition 19. *A dilation δ defined by Equation 24 is increasing with respect to the bipolar fuzzy structuring element, while an erosion ε defined by Equation 23 is decreasing with respect to the bipolar fuzzy structuring element.*

Proof. The results are directly derived from the increasingness of $C, \vee, \wedge$ and from the decreasingness of I with respect to the first argument. $\square$

These results fit well with the intuitive meaning behind the morphological operators. Indeed, a dilation is interpreted as a degree of intersection, which is easier to achieve with a larger structuring element, while an erosion is interpreted as a degree of inclusion, which means a stronger constraint if the structuring element is larger.

Proposition 20. *Let δ and ε be a dilation and an erosion defined by Equations 24 and 23. Then, for all $(\mu_B, \nu_B), (\mu'_B, \nu'_B), (\mu, \nu)$ in $\mathcal{B}$, we have:*

$$\delta_{(\mu_B,\nu_B)\wedge(\mu'_B,\nu'_B)}((\mu,\nu) \preceq \delta_{(\mu_B,\nu_B)}((\mu,\nu)) \wedge \delta_{(\mu'_B,\nu'_B)}((\mu,\nu)), \tag{27}$$

$$\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \vee \varepsilon_{(\mu'_B,\nu'_B)}((\mu,\nu)) \preceq \varepsilon_{(\mu_B,\nu_B)\wedge(\mu'_B,\nu'_B)}((\mu,\nu)). \tag{28}$$

Proof. The results are derived from the increasingness of C and the decreasingness of I with respect to the first argument. $\square$

Depending on the choice of C and I, some additional property may hold.

Proposition 21. *Let δ be a dilation defined by Equation 24 from a bipolar conjunction C. The dilation satisfies $\delta_{(\mu_B,\nu_B)}((\mu,\nu)) = \delta_{(\mu,\nu)}((\mu_B,\nu_B))$ if and only if C is commutative.*

Proof. The implication directly results from the commutativity of C. The converse implication is obtained by considering constant membership and non-membership functions for the bipolar fuzzy sets. $\square$

This result is quite intuitive. When interpreting the dilation as a degree of intersection, it is natural to expect this degree to be symmetrical in both arguments. Hence the commutativity of C has to be satisfied.

Proposition 22. *Let δ be a dilation defined by Equation 24 from a bipolar conjunction C. It satisfies the iterativity property, i.e.:*

$$\delta_{(\mu_B,\nu_B)}(\delta_{(\mu'_B,\nu'_B)}(\mu,\nu)) = \delta_{\delta_{(\mu_B,\nu_B)}((\mu'_B,\nu'_B))}((\mu,\nu)),$$

if and only if C is associative.

Proof. Similar to the one of Proposition 21. $\square$

Proposition 23. *Let δ be a dilation defined by Equation 24 from a bipolar conjunction C. If C is a bipolar conjunction that admits $1_{\mathcal{L}}$ as unit element on the left (i.e. $\forall (a,b) \in \mathcal{L}, C(1_{\mathcal{L}},(a,b)) = (a,b)$) and $C((a,b),1_{\mathcal{L}}) \neq 1_{\mathcal{L}}$ for $(a,b) \neq 1_{\mathcal{L}}$, then the dilation is extensive, i.e. $\delta_{(\mu_B,\nu_B)}((\mu,\nu)) \succeq (\mu,\nu)$, if and only if $(\mu_B,\nu_B)(0) = 1_{\mathcal{L}}$, where 0 denotes the origin of space $\mathcal{S}$.*

A similar property holds for erosion and if I is a bipolar implication that admits $1_{\mathcal{L}}$ as unit element to the left (i.e. $\forall (a,b) \in \mathcal{L}, I(1_{\mathcal{L}},(a,b)) = (a,b)$) and $I((a,b),0_{\mathcal{L}}) \neq 0_{\mathcal{L}}$ for $(a,b) \neq 1_{\mathcal{L}}$, then the erosion is anti-extensive, i.e. $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \preceq (\mu,\nu)$, if and only if $(\mu_B,\nu_B)(0) = 1_{\mathcal{L}}$.

Proof. Let C be a conjunction satisfying the conditions, and assume that $(\mu_B,\nu_B)(0) = 1_{\mathcal{L}}$. Then, $\forall (\mu_B,\nu_B) \in \mathcal{B}, (\mu,\nu) \in \mathcal{B}, \forall x \in \mathcal{S}$, since $1_{\mathcal{L}}$ is unit element on the left we have:

$$\begin{aligned}
\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) &\succeq C((\mu_B,\nu_B)(0),(\mu,\nu)(x)) \\
&\succeq C(1_{\mathcal{L}},(\mu,\nu)(x)) \\
&\succeq (\mu,\nu)(x)
\end{aligned}$$

i.e. δ is extensive.

Conversely, if δ is extensive, let us write the extensivity inequality for the bipolar fuzzy set (μ,ν) defined by $(\mu,\nu)(y) = 0_{\mathcal{L}}$ for $y \neq 0$ and $(\mu,\nu)(0) = 1_{\mathcal{L}}$ and for $x = 0_{\mathcal{L}}$:

$$\vee_y C((\mu_B,\nu_B)(-y),(\mu,\nu)(y)) \succeq (\mu,\nu)(0)$$

$$\Rightarrow \vee_{y\neq 0} C((\mu_B,\nu_B)(-y),0_{\mathcal{L}}) \vee C((\mu_B,\nu_B)(0),1_{\mathcal{L}}) = 1_{\mathcal{L}}$$

$$\Rightarrow 0_{\mathcal{L}} \vee C((\mu_B,\nu_B)(0),1_{\mathcal{L}}) = C((\mu_B,\nu_B)(0),1_{\mathcal{L}}) = 1_{\mathcal{L}}$$

(from Proposition 2). Since under the hypothesis the only value of (a,b) for which $C((a,b),1_{\mathcal{L}})$ can be equal to $1_{\mathcal{L}}$ is $1_{\mathcal{L}}$, it follows that $(\mu_B,\nu_B)(0) = 1_{\mathcal{L}}$.

The proof for I and ε is similar. □

The second condition on C holds in particular if $1_{\mathcal{L}}$ is also unit element on the right. This holds in specific cases in which C is a bipolar t-norm, which are the most interesting ones from a morphological point of view, as shown below.

Note that the condition $(\mu_B,\nu_B)(0) = 1_{\mathcal{L}}$ (i.e. the origin of space completely belongs to the bipolar fuzzy set, without any indetermination) is equivalent to the conditions on the structuring element found in classical [88] and fuzzy [30] morphology to have extensive dilations and anti-extensive erosions.

Proposition 24. *If I is derived from C and a negation N, then δ and ε are dual operators, i.e.: $\delta_{(\mu_B,\nu_B)}(N(\mu,\nu)) = N(\varepsilon_{(\check{\mu}_B,\check{\nu}_B)}((\mu,\nu)))$, where $(\check{\mu}_B,\check{\nu}_B)$ denotes the symmetrical of (μ_B,ν_B) with respect to the origin of $\mathcal{S}$.*

Proof. This result directly follows from Equation 7. □

Duality with respect to complementation, which was advocated in the first developments of mathematical morphology [88], is important to handle in an consistent way an object and its complement for many applications (for instance in image processing and spatial reasoning). Therefore it is useful to know exactly under which conditions this property may hold, so as to choose the appropriate operators if it is needed for a specific problem. On the other hand, adjunction is a major feature of the "modern" view of mathematical morphology, with strong algebraic bases in the framework of complete lattices [84]. This framework is now widely considered as the most interesting one, since it provides consistent definitions with sound properties in different settings (continuous and discrete ones) and extending mathematical morphology to fuzzy sets in this framework inherits a set of powerful and important properties. Due to the interesting features of these two properties of duality and adjunction, in several applications both are required.

From all these results, we can derive the following theorem, which shows that the proposed forms are the most general ones for C being a bipolar t-norm.

Theorem 1. *Definition 7 defines a dilation with all properties of classical mathematical morphology if and only if C is a bipolar t-norm. The adjoint erosion is then defined by Equation 6 from the residual implication I_R derived from C. If the duality property is additionally required, then C and I have also to be dual operators with respect to a negation N.*

Proof. This theorem directly follows from the previous propositions. □

This important result shows that taking any conjunction may not lead to dilations that have nice properties. For instance the iterativity of dilation is of prime importance in concrete applications, and it requires associative conjunctions.

This is actually a main contribution of our work, which differs from [79], where some morphological operators are suggested on intuitionistic fuzzy sets and for the Pareto ordering, but without referring to the algebraic framework, and leading to weaker properties (for instance the erosion defined in this work does not commute with the infimum and is then not an algebraic erosion). This group has then proposed some extensions in [74], still for the specific case of Pareto ordering, which closely follow our previous results in [19, 21, 24, 26]. Moreover the result expressed in Theorem 1 is stronger and more general since it applies for any partial ordering leading to a complete lattice on $\mathcal{B}$.

Note that pairs of adjoint operators are not necessarily dual. Therefore requiring both adjunction and duality properties may drastically reduce the choice for C and I. This will be illustrated for $\preceq$ being the Pareto partial ordering in Section 5. Note that this strong constraint is similar to the one proved for fuzzy sets in [17, 22].

Although the choice of C and I is limited by the results expressed in Theorem 1 if sufficiently strong properties are required for the morphological operators, some choice may remain. The following property expresses a monotony property with respect to this choice.

Proposition 25. *Dilations and erosions are monotonous with respect to the choice of C and I:*

$$C \preceq C' \Rightarrow \delta^C \preceq \delta^{C'}$$

where δ^C is the dilation defined by Equation 24 using the bipolar conjunction or t-norm C, and

$$I \preceq I' \Rightarrow \varepsilon^I \preceq \varepsilon^{I'}$$

where ε^I is the erosion defined by Equation 23 using the bipolar implication I.

Proof. The result is straightforward, from the monotony of the supremum and infimum. $\square$

More properties on the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are provided in Section 7.3.

5 Pareto (marginal) partial ordering

In this section, we detail the case of Pareto ordering, in order to illustrate the general definitions and results of Sections 3 and 4. This summarizes our previous results in [19, 21, 24, 26], with some extensions.

5.1 Complete lattice derived from Pareto ordering and connectives

The marginal partial ordering on $\mathcal{L}$, or Pareto ordering (by reversing the scale of negative information) is defined as:

$$(a_1, b_1) \preceq_p (a_2, b_2) \text{ iff } a_1 \leq a_2 \text{ and } b_1 \geq b_2. \tag{29}$$

This ordering, often used in economics and social choice, has been been used for bipolar information [59], intuitionistic fuzzy sets e.g. in [45], or interval-valued fuzzy sets [74].

For this partial ordering, $(\mathcal{L}, \preceq_p)$ is a complete lattice. The greatest element is $(1, 0)$ and the smallest element is $(0, 1)$. The supremum and infimum are respectively defined as:

$$(a_1, b_1) \vee_p (a_2, b_2) = (\max(a_1, a_2), \min(b_1, b_2)), \tag{30}$$

$$(a_1, b_1) \wedge_p (a_2, b_2) = (\min(a_1, a_2), \max(b_1, b_2)). \tag{31}$$

The partial order $\preceq_p$ induces a partial order on the set of bipolar fuzzy sets:

Definition 8. *A Pareto ordering on $\mathcal{B}$ is defined as:*
$\forall (\mu_1, \nu_1) \in \mathcal{B}, \forall (\mu_2, \nu_2) \in \mathcal{B},$

$$(\mu_1, \nu_1) \preceq_p (\mu_2, \nu_2) \textit{ iff } \forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x) \textit{ and } \nu_1(x) \geq \nu_2(x). \tag{32}$$

Note that this corresponds formally to the inclusion on intuitionistic fuzzy sets [6] (again the semantics are different).

Proposition 26. $(\mathcal{B}, \preceq_p)$ *is a complete lattice. The supremum and the infimum are:*

$$\forall x \in \mathcal{S}, ((\mu_1, \nu_1) \vee_p (\mu_2, \nu_2))(x) = (\max(\mu_1(x), \mu_2(x)), \min(\nu_1(x), \nu_2(x))), \tag{33}$$

$$\forall x \in \mathcal{S}, ((\mu_1, \nu_1) \wedge_p (\mu_2, \nu_2))(x) = (\min(\mu_1(x), \mu_2(x)), \max(\nu_1(x), \nu_2(x))). \tag{34}$$

Let us now consider any family of bipolar fuzzy sets $(\mu_i, \nu_i), i \in I$, where the index set I can be finite or not. Supremum and infimum for any family are expressed similarly:

$$\forall x \in \mathcal{S}, \bigvee{}_p{}_{i \in I} (\mu_i, \nu_i)(x) = (\sup_{i \in I} \mu_i(x), \inf_{i \in I} \nu_i(x)),$$

$$\forall x \in \mathcal{S}, \bigwedge{}_p{}_{i \in I} (\mu_i, \nu_i)(x) = (\inf_{i \in I} \mu_i(x), \sup_{i \in I} \nu_i(x)),$$

The greatest element is the pair of functions $(\mu_{\mathbb{I}}, \nu_{\mathbb{I}})$ constantly equal $1_{\mathcal{L}}$, and the smallest element is the pair of functions (μ_0, ν_0) constantly equal to $0_{\mathcal{L}}$.

Let us now mention a few connectives. In Definition 2, the monotony properties have now to be intended according to the Pareto ordering.

An example of negation, which will be used in the following, is the standard negation, defined by $N((a,b)) = (b,a)$.

Two types of t-norms and t-conorms are considered in [51] (actually in the intuitionistic case) and will be considered here as well in the bipolar case:

1. Operators called t-representable bipolar t-norms and t-conorms, which can be expressed using usual t-norms t and t-conorms T:

$$C((a_1,b_1),(a_2,b_2)) = (t(a_1,a_2), T(b_1,b_2)), \tag{35}$$

$$D((a_1,b_1),(a_2,b_2)) = (T(a_1,a_2), t(b_1,b_2)). \tag{36}$$

   A typical example is obtained for $t = \min$ and $T = \max$. Although t and T are usually chosen as dual operators, other choices are possible, as discussed e.g. in [73] for adjunction properties. Distributivity properties of implications over t-norms are further investigated in [8].

2. Bipolar Lukasiewicz operators, which are not t-representable:

$$C_W((a_1,b_1),(a_2,b_2)) = (\max(0, a_1+a_2-1), \min(1, b_1+1-a_2, b_2+1-a_1)), \tag{37}$$

$$D_W((a_1,b_1),(a_2,b_2)) = (\min(1, a_1+1-b_2, a_2+1-b_1), \max(0, b_1+b_2-1)). \tag{38}$$

In these equations, the positive part of C_W is the usual Lukasiewicz t-norm of a_1 and a_2 (i.e. the positive parts of the input bipolar values). The negative part of D_W is the usual Lukasiewicz t-norm of the negative parts (b_1 and b_2) of the input values.

The two types of implication introduced in Section 3 can be used here as well, and were also considered in [44, 51]. The two types of implication coincide for the Lukasiewicz operators [45].

Proposition 27. *Let us denote by C_{min} (respectively D_{max}) the t-representable bipolar conjunction (respectively disjunction) built from the minimum and maximum, and C_{prod} (respectively D_{sum}) the one built from the product and algebraic sum. We have the following ordering between conjunctions: $\forall((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,*

$$C_W((a_1,b_1),(a_2,b_2)) \preceq_p C_{prod}((a_1,b_1),(a_2,b_2)) \preceq_p C_{min}((a_1,b_1),(a_2,b_2)), \tag{39}$$

and for disjunctions:

$$D_{max}((a_1,b_1),(a_2,b_2)) \preceq_p D_{sum}((a_1,b_1),(a_2,b_2)) \preceq_p D_W((a_1,b_1),(a_2,b_2)). \tag{40}$$

Proof. In the fuzzy (non-bipolar case), the Lukasiewicz t-norm is smaller than the product which is smaller than the minimum (largest t-norm). Therefore the inequalities between t-representable bipolar t-norms are straightforward, as well as the one with C_W for its positive part. It is then enough to show the inequality for the negative part of C_W. We have $1 - a_1 \geq b_1$ and $1 - a_2 \geq b_2$ hence $b_2 + 1 - a_1 \geq b_2 + b_1$ and $b_1 + 1 - a_2 \geq b_1 + b_2$. Therefore the negative part of C_W is larger than $\min(1, b_1 + b_2)$, which completes the proof for bipolar t-norms. The reasoning for bipolar t-conorms follows the same line. □

Morphological operators derived from these connectives will inherit these properties.

5.2 Algebraic and morphological erosions and dilations

Since the Pareto ordering is an example leading to complete lattices, algebraic dilations and erosions can be defined as in Section 3.

Next, introducing structuring elements, morphological erosions and dilations are defined as in Equations 23 and 24.

It is easy to show that the bipolar Lukasiewicz operators are adjoint, according to Equation 4. Therefore, if Lukasiewicz operators (up to a bijection) are used, then all algebraic properties detailed in Section 3 hold.

Moreover, it has been shown that the adjoint operators are all derived from the Lukasiewicz operator, using a continuous bijective permutation on $[0, 1]$ [51]. Hence having dilations and erosions that are both dual and adjoint can be achieved only for this class of operators. This completes the result of Theorem 1 in the particular case of the Pareto ordering.

If the bipolar fuzzy sets are usual fuzzy sets (i.e. $\nu = 1-\mu$ and $\nu_B = 1-\mu_B$), the definitions based on the bipolar Lukasiewicz operators are equivalent to fuzzy dilation and erosion defined for the classical Lukasiewicz t-norm and t-conorm [22, 30].

Details can be found in [19, 21, 26].

5.3 Interpretation

In order to interpret the expression of morphological erosion, let us first consider the implication I defined from a t-representable bipolar t-conorm D, i.e. $\forall((a,b),(a',b')) \in \mathcal{L}^2, I((a,b),(a',b')) = D((b,a),(a',b'))$, when using the standard negation, and D is defined as in Equation 36. Then the erosion writes:

$$\begin{aligned}
\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) &= \bigwedge\nolimits_{p}{}_{\,y\in\mathcal{S}} I((\mu_B(y-x),\nu_B(y-x)),(\mu(y),\nu(y))) \\
&= \bigwedge\nolimits_{p}{}_{\,y\in\mathcal{S}} (T((\nu_B(y-x),\mu(y)), t(\mu_B(y-x),\nu(y))) \\
&= (\inf_{y\in\mathcal{S}} T((\nu_B(y-x),\mu(y)), \sup_{y\in\mathcal{S}} t(\mu_B(y-x),\nu(y))) \qquad (41)
\end{aligned}$$

The second line is derived from the fact that D is supposed here to be a t-representable bipolar t-conorm, defined from a t-norm t and a t-conorm T. The third line is derived from the definition of the infimum in $\mathcal{L}$ and in $\mathcal{B}$ for $\preceq_p$. This resulting bipolar fuzzy set has a membership function which is exactly the fuzzy erosion of μ by the fuzzy structuring element $1-\nu_B$, according to the original definitions in the fuzzy case [30]. The non-membership function is exactly the dilation of the fuzzy set ν by the fuzzy structuring element μ_B.

Let us consider the dilation, defined from a t-representable t-norm C (Equation 35). Using the standard negation, it writes:

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = (\sup_{y\in\mathcal{S}} t(\mu_B(x-y),\mu(y)), \inf_{y\in\mathcal{S}} T((\nu_B(x-y),\nu(y))). \quad (42)$$

The first term (membership function) is exactly the fuzzy dilation of μ by μ_B, while the second one (non-membership function) is the fuzzy erosion of ν by $1-\nu_B$, according to the original definitions in the fuzzy case [30].

This observation has a nice interpretation, which well fits with intuition. Let (μ,ν) represent a spatial bipolar fuzzy set, where μ is a positive information for the location of an object for instance, and ν a negative information for this location. A bipolar structuring element can represent additional imprecision on the location, or additional possible locations. Dilating (μ,ν) by this bipolar structuring element amounts to dilate μ by μ_B, i.e. the positive region is extended by an amount represented by the positive information encoded in the structuring element. On the contrary, the negative information is eroded by the complement of the negative information encoded in the structuring element. This corresponds well to what would be intuitively expected in such situations. A similar interpretation can be provided for the bipolar fuzzy erosion. Examples are provided in the next subsections.

Let us now consider the implication derived from the Lukasiewicz bipolar operators (Equations 37 and 38). The erosion and dilation write:

$$\begin{aligned}&\forall x\in\mathcal{S}, \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \\ &\bigwedge\nolimits^{p}{}_{y\in\mathcal{S}} (\min(1,\mu(y)+1-\mu_B(y-x),\nu_B(y-x)+1-\nu(y)), \max(0,\nu(y)+\mu_B(y-x)-1)) = \\ &(\inf_{y\in\mathcal{S}} \min(1,\mu(y)+1-\mu_B(y-x),\nu_B(y-x)+1-\nu(y)), \sup_{y\in\mathcal{S}} \max(0,\nu(y)+\mu_B(y-x)-1)),\end{aligned} \quad (43)$$

$$\begin{aligned}&\forall x\in\mathcal{S}, \delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \\ &(\sup_{y\in\mathcal{S}} \max(0,\mu(y)+\mu_B(x-y)-1), \inf_{y\in\mathcal{S}} \min(1,\nu(y)+1-\mu_B(x-y),\nu_B(x-y)+1-\mu(y)).\end{aligned} \quad (44)$$

The negative part of the erosion is exactly the fuzzy dilation of ν (negative part of the input bipolar fuzzy set) with the structuring element μ_B (positive part of the bipolar fuzzy structuring element), using the Lukasiewicz t-norm. Similarly, the positive part of the dilation is the fuzzy dilation of μ (positive part of the input) by μ_B (positive part of the bipolar fuzzy structuring element), using the Lukasiewicz t-norm. Hence for both operators, the "dilation" part (i.e. negative part for the erosion and positive part for the dilation) has always a direct interpretation and is the same as the one obtained using t-representable operators, for t being the Lukasiewicz t-norm.

In the case the structuring element is non bipolar (i.e. $\forall x \in \mathcal{S}, \nu_B(x) = 1 - \mu_B(x)$), then the "erosion" part has also a direct interpretation: the positive part of the erosion is the fuzzy erosion of μ by μ_B for the Lukasiewicz t-conorm; the negative part of the dilation is the erosion of ν by μ_B for the Lukasiewicz t-conorm.

It follows from Propositions 25 and 27 that the some erosions and dilations can be ordered according to the used connectives.

Proposition 28. *Let us denote by δ^{min}, δ^{prod} and δ^W the dilations built from C_{min}, C_{prod} and C_W, respectively. We have the following ordering: $\forall((\mu_B, \nu_B), (\mu, \nu)) \in \mathcal{B}^2$,*

$$\delta^W_{(\mu_B,\nu_B)}(\mu, \nu) \preceq_p \delta^{prod}_{(\mu_B,\nu_B)}(\mu, \nu) \preceq_p \delta^{min}_{(\mu_B,\nu_B)}(\mu, \nu). \tag{45}$$

Let us denote by ε^{max}, ε^{sum} and ε^W the erosions built from the implications derived from D_{max}, D_{sum} and D_W, respectively. We have the following ordering: $\forall((\mu_B, \nu_B), (\mu, \nu)) \in \mathcal{B}^2$,

$$\varepsilon^{max}_{(\mu_B,\nu_B)}(\mu, \nu) \preceq_p \varepsilon^{sum}_{(\mu_B,\nu_B)}(\mu, \nu) \preceq_p \varepsilon^{W}_{(\mu_B,\nu_B)}(\mu, \nu). \tag{46}$$

This means that operations built from min and max have a stronger effect on the initial bipolar fuzzy set.

Let us finally comment on the practical use of these operators, where discretization may induce some approximations. This has already been addressed in the case of interval-valued fuzzy sets in [75]. The discretization of the space $\mathcal{S}$ does not induce any particular problem. As for the values of μ and ν, the discretization of $[0, 1]$ may induce some small errors depending on the choice of C and I. Let us assume that the values are regularly discretized (as is usually the case), in the form $\frac{k}{n}$ where k and n are integer values, with n defining the granularity of the discretization and $0 \leq k \leq n$. The negation, minimum, maximum and Lukasiewicz operators provide exact results, and hence $C_{min}, C_W, D_{max}, D_W, \delta^{min}, \delta^W, \varepsilon^{max}, \varepsilon^W$. However the product and algebraic sum (and thus $C_{prod}, D_{sum}, \delta^{prod}, \varepsilon^{sum}$) need some approximation. For a quantification on 6 to 10 bits ($n = 2^6 - 1$ to $n = 2^{10} - 1$), we have tested that the maximal error on the product does not exceed the quantification step $\frac{1}{n}$. Therefore the approximation errors can be considered as low enough to be neglected in the applications.

5.4 Illustrative example in the spatial domain

When dealing with spatial information, in image processing or for spatial reasoning applications, bipolarity may be an important feature of the information to be processed. For instance, when assessing the position of an object in space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible or forbidden places (for instance because they are occupied by other objects). As another example, let us consider spatial relations. Human beings consider "left" and "right" as opposite relations. But this does not mean that one of them is the negation of the other one. The semantics of "opposite" captures a notion of symmetry (with respect to some axis or plane) rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving room for some indetermination [13]. This corresponds to the idea that the union of positive and negative information does not cover all the space. Similar considerations can be provided for other pairs of "opposite" relations, such as "close to" and "far from" for instance.

An example is illustrated in Figure 1. It shows an object at some position in the space (the rectangle in this figure). For visualization purposes, in all illustrations, a representation using grey levels is adopted for encoding $\mu(x)$ and $\nu(x)$ (0 = black, 1 = white). Two images are shown, one for positive information and one for negative information.

Let us assume that some information about the position of another object is provided: it is to the left of the rectangle and not to the right. The region "to the left of the rectangle" is computed using a fuzzy dilation with a directional fuzzy structuring element providing the semantics of "to the left" [13], thus defining the positive information. The region "to the right of the rectangle" defines the negative information and is computed in a similar way. The membership functions μ_L and μ_R represent respectively the positive and negative parts of the bipolar fuzzy set. They are not the complement of each other, and we have: $\forall x, \mu_L(x) + \mu_R(x) \leq 1$. Here we assume that this consistency constraints hold. A discussion on how to achieve it can be found in [25].

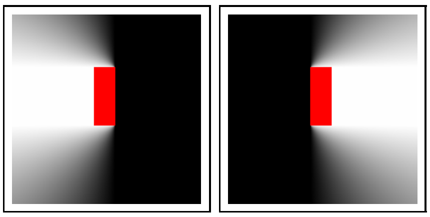


Figure 1: Region to the left of the rectangle (positive information, μ_L) and region to the right of the rectangle (negative information, μ_R). The membership degrees vary from 0 (black) to 1 (white).

Another example, for the pair of relations close/far, is illustrated in Figure 2. The reference object is the square in the center of the image. The two fuzzy regions are computed using fuzzy dilations, using structuring elements that provide the semantics of "close" and "far" [14]. Again, the two membership functions μ_C and μ_F are not the complement of each other and actually define a bipolar fuzzy set, with its positive and negative parts.

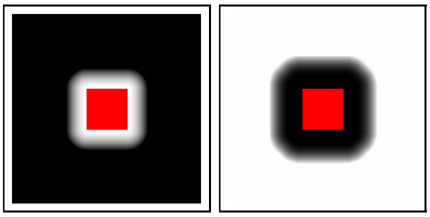


Figure 2: Region close to the square (μ_C) and region far from the square (μ_F).

Note that considering "opposite" relations is but one example. Other examples could be provided, where relations could not be opposite, or even of different nature. For instance: we have some positive information for an object being above a reference object (directional relation), and some negative information for the object not being in some region of space (topological relation).

To our knowledge, bipolarity has not been much exploited in the spatial domain. A few works deal with image thresholding, filtering, or edge detection, based on intuitionistic fuzzy sets derived from image intensity and entropy or divergence criteria [12, 40, 47, 97]. Spatial representations of interval-valued fuzzy sets have also been proposed in [41], as a kind of fuzzy egg-yolk, for evaluating classification errors based on ground-truth, or in [70, 71] with preliminary extensions of RCC to these representations. But there are still very few tools for manipulating spatial information using both its bipolarity and imprecision components.

Let us now illustrate the proposed morphological operations on the simple example shown in Figure 1. Let us assume that an additional information, given as a bipolar structuring element, allows us to reduce the positive part and to extend the negative part of the bipolar fuzzy region. This can be formally expressed as a bipolar fuzzy erosion, applied to the bipolar fuzzy set (μ_L, μ_R), using this structuring element. Figure 3 illustrates the result. It can be observed that the region corresponding to the positive information has actually been reduced (via a fuzzy erosion), while the region corresponding to the negative part has been extended (via a fuzzy dilation).

An example of bipolar fuzzy dilation is illustrated in Figure 4 for the bipolar fuzzy set close/far of Figure 2. The dilation corresponds to a situation where the structuring element represents by how much the positive part of the information can be expanded (positive part of the structuring element), for instance because new positions become possible, and by how much the negative part of the information should be reduced (negative part of the structuring element), for instance because it was too severe. These operations allow modifying the semantics attached to the concepts "close" and "far": in this example, a larger space around the object is considered being close to the object, and the regions that are considered being far from the object are put further away.

When several pieces of information are available, such as information on direction and information on distance, they can be combined using fusion tools, in order to get a spatial region accounting for all available information. This

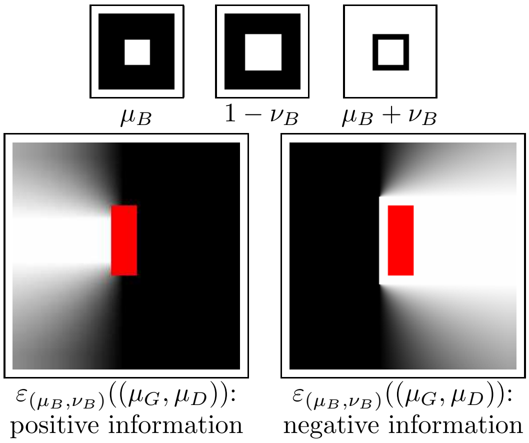


Figure 3: Illustration of a bipolar fuzzy erosion on the example shown in Figure 1. The results, displayed on the second line, show the reduction of the positive part the extension of the negative part.

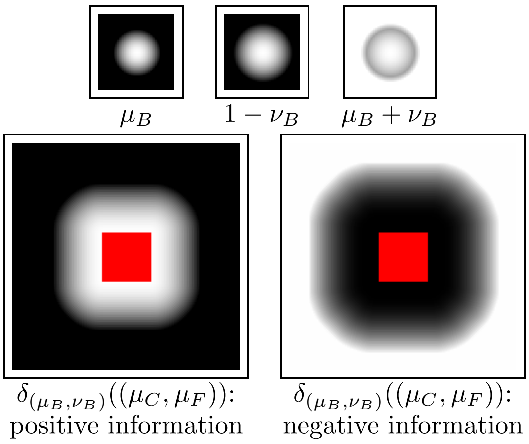


Figure 4: Illustration of a bipolar fuzzy dilation on the example shown in Figure 2. The results show the extension of the positive part and the reduction of the negative part.

type of approach has been used to guide the recognition of anatomical structures in images, based on medical knowledge expressed as a set of spatial relations between pairs or triplets of structures (e.g. in an ontology), in the fuzzy case [27, 43, 64]. This idea can be extended to the bipolar case. As an example, a result of fusion of directional and distance information is illustrated in Figure 5. The positive information "to the left" of the reference object (and the negative part "to the right") is combined with the dilated distance information shown in Figure 4. The positive parts are combined in a conjunctive way (using a min here) and the negative parts in a disjunctive way (as a max here), according to the semantics of the fusion of bipolar information [55]. This example shows how the search space can be reduced by combining spatial relations to reference objects, expressed as bipolar fuzzy sets. This can be considered as an extension to the bipolar case of attention focusing approaches. Illustrations of this idea on the problem of recognition of brain structures from magnetic resonance imaging are presented in [21, 25], and the integration of bipolar fuzzy mathematical morphology into descriptions logics for spatial reasoning has been proposed in [65].

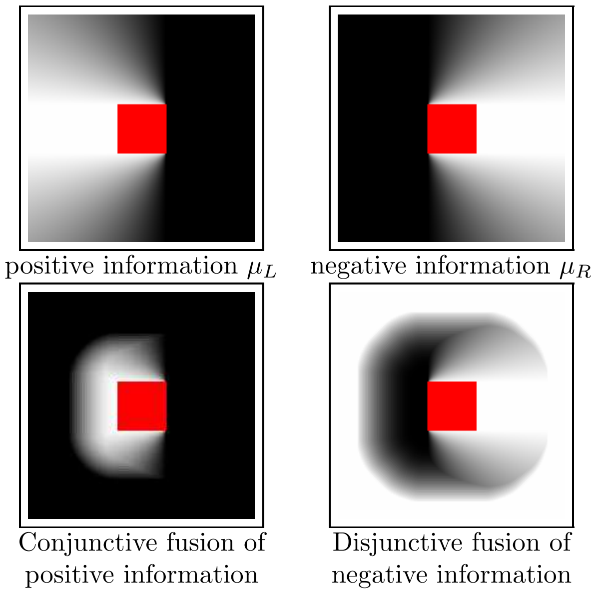


Figure 5: Fusion of bipolar information on direction (μ_L, μ_R) and on distance $\delta_{(\mu_B, \nu'_B)}((\mu_C, \mu_F))$ of Figure 4.

5.5 Example in preference modeling

To illustrate the features of bipolar mathematical morphology in another domain, we propose here a simple example in a logical formalism, as briefly introduced in Section 2.1.

Let us consider preferences of agents about the countries in which they would like to travel, and constraints about their travels. The set of propositional symbols if the set of all countries in the world. Preferences are denoted by formulas φ and constraints by formulas ψ. In the following example, we show how dilation of bipolar representations of preferences and constraints can help reaching an agreement between agents.

Let us assume that Agent 1:

- prefers to travel in Spain: $\varphi_1 = Spain$,
- has to stay in Europe: $\psi_1 = \neg(Belgium \vee France \vee Spain \vee Portugal \vee Italy \vee Germany \vee TheNetherlands \vee ...\}$.

On the other hand, Agent 2:

- prefers to travel in Morocco: $\varphi_2 = Morocco$,
- has to stay in a Mediterranean country: $\psi_2 = \neg(Morocco \vee Spain \vee Italy \vee Portugal \vee ...)$.

In this example, the two agents have conflicting preferences. However, each agent is now ready to extend his preferences so that the two agents can travel together (under the conditions that the constraints, which are fixed, are satisfied). This is can be simply modeled by a dilation such that some neighbor countries are included in the preferences, conditioned by the constraints:

$$\delta(\varphi_1) = Spain \vee France \vee Portugal \vee Morocco$$

$$\delta(\varphi_2) = Morocco \vee Algeria \vee Portugal \vee Spain$$

Introducing the constraints in order to satisfy the consistency requirements leads to:

$$\varphi_1' = \delta(\varphi_1) \wedge \psi_1 = Spain \vee France \vee Portugal$$

$$\varphi_2' = \delta(\varphi_2) \wedge \psi_2 = \delta(\varphi_2)$$

Now the preferences are no more conflicting. The fusion of their preferences and constraints can be expressed as the conjunction of the preferences and disjunction of the constraints:

$$\begin{aligned}(\varphi, \psi) &= (\varphi_1' \wedge \varphi_2', \psi_1 \vee \psi_2) \\ &= (Spain \vee Portugal, \neg(\bigvee Medit.\, and\, Eur.\, countries))\end{aligned}$$

A solution for travelling can then be found in the set of models of these formulas.

6 Bipolar fuzzy mathematical morphology based on lexicographic ordering

In the previous section, we developed a theory for mathematical morphology on bipolar fuzzy sets based on Pareto partial ordering. This implies that positive information and negative information play symmetrical roles. However, based on the discussion about semantics in Section 2.4, this might not always be appropriate, since we may want to process positive and negative information in different ways, in particular when the two types of information are issued from different sources or have different semantics. For instance if the positive information represents preferences and the negative information rules or constraints, then it may be interesting (or mandatory) to give more priority to the constraints (or in the contrary to the positive information) [9, 10, 57, 66, 82]. The partial ordering should then be replaced by another one, accounting for these priorities. Moreover, in the particular context of mathematical morphology, this ordering has an additional drawback: the value at a point in the resulting dilation or erosion is generally expected to be one of the values of neighborhood points (defined by the structuring element), but this is in general not the case when using Pareto ordering. This point has already raised discussions in the mathematical morphology community, in particular when dealing with vector-valued images, such as color images (see e.g. [3, 5, 95]). It has been shown that non vector-preserving orderings may lead to counter-intuitive results (for instance introducing new colors, that do not belong to any of the image objects, may prevent their correct recognition).

In this section, we therefore introduce priorities between the two types of information, based on a lexicographic ordering which induces another way of modeling mathematical morphology, and which guarantees that the resulting bipolar value at a point is one of the values of neighborhood points. Thus, this addresses the two issues mentioned above. The lexicographic ordering (also called dictionary ordering) is denoted by $\preceq_L$. It is additionally a total order on $\mathcal{L}$. On the induced lattice on $\mathcal{B}$, we define algebraic dilations and erosions. We also propose connectives that are adapted to this ordering, and then derive morphological dilations and erosions. This extends a preliminary work in [26].

6.1 Lexicographic ordering and associated lattice

Definition 9. *The lexicographic relation $\preceq_L$ on $\mathcal{L}$, giving priority to negative information, is defined as:*

$$(a, b) \preceq_L (a', b') \Leftrightarrow b > b' \text{ or } (b = b' \text{ and } a \leq a'). \tag{47}$$

Proposition 29. *The relation $\preceq_L$ defines a total ordering on $\mathcal{L}$ and $(\mathcal{L}, \preceq_L)$ is a complete lattice. The smallest element is $(0, 1)$ and the largest element is $(1, 0)$.*

A lexicographic ordering giving priority to the positive information can be defined in a similar way. All what follows applies in both cases, and we only detail the one of Definition 9 in this paper.

Figure 6 illustrates the difference between $\preceq_p$ and $\preceq_L$.

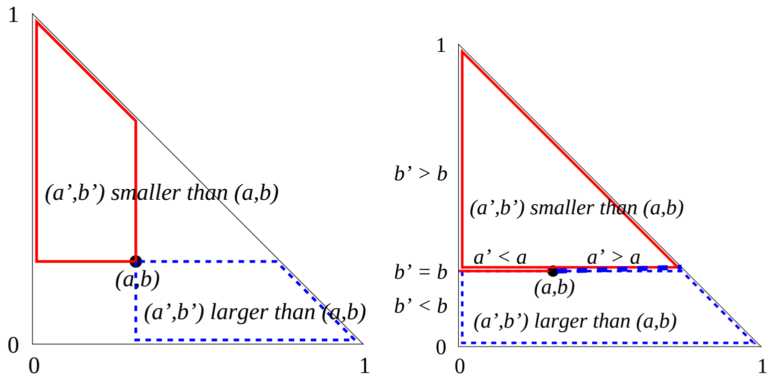


Figure 6: Comparison, in $\mathcal{L}$, between the partial ordering $\preceq_p$ (left) and the total ordering $\preceq_L$ (right). Plain (respectively dashed) lines indicate the regions of $\mathcal{L}$ in which points (a', b') are smaller (respectively larger) than point (a, b).

This ordering induces a partial ordering on $\mathcal{B}$ (the same notation is used):

Definition 10. *The lexicographic relation on $\mathcal{B}$ is defined by:*

$$(\mu, \nu) \preceq_L (\mu', \nu') \Leftrightarrow \forall x \in \mathcal{S}, (\mu(x), \nu(x)) \preceq_L (\mu'(x), \nu'(x)). \tag{48}$$

This definition means that a bipolar fuzzy set is considered as smaller than another one if its negative part is larger, or if the two negative parts are equal and the positive part is smaller. This strongly expresses the priority given to the negative information, since only the negative parts are considered as soon as they differ.

Proposition 30. *The relation $\preceq_L$ (Definition 10) defines a partial ordering, called lexicographic ordering, on $\mathcal{B}$ and $(\mathcal{B}, \preceq_L)$ is a complete lattice. The smallest element is (μ_0, ν_0) (defined by $\forall x \in \mathcal{S}, \mu_0(x) = 0, \nu_0(x) = 1$), and the largest element is $(\mu_{\mathbb{I}}, \nu_{\mathbb{I}})$ (defined by $\forall x \in \mathcal{S}, \mu_{\mathbb{I}}(x) = 1, \nu_{\mathbb{I}}(x) = 0$).*

Proposition 31. *Infimum and supremum for $\preceq_L$ are expressed, for any two elements (a, b) and (a', b') of $\mathcal{L}$, as:*

$$\min{}_{\preceq_L}((a, b), (a', b')) = \begin{cases} (a, b) & \text{if } b > b' \\ (a', b') & \text{if } b < b' \\ (\min(a, a'), b) & \text{if } b = b' \end{cases} \tag{49}$$

$$\max{}_{\preceq_L}((a, b), (a', b')) = \begin{cases} (a, b) & \text{if } b < b' \\ (a', b') & \text{if } b > b' \\ (\max(a, a'), b) & \text{if } b = b' \end{cases} \tag{50}$$

Infimum and supremum for any family of elements of $\mathcal{L}$ or $\mathcal{B}$ are derived in a straightforward way, and are denoted by $\bigwedge_{\preceq_L}$ and $\bigvee_{\preceq_L}$. They can be computed using fast sorting algorithms.

Let us note that, in all cases, the lexicographic minimum (or maximum) provides a result which is one of the input bipolar values, and the following equivalences hold:

$$\min{}_{\preceq_L}((a,b),(a',b')) = (a,b) \Leftrightarrow (a,b) \preceq_L (a',b'),$$

$$\max{}_{\preceq_L}((a,b),(a',b')) = (a,b) \Leftrightarrow (a,b) \succeq_L (a',b').$$

6.2 Connectives

Bipolar connectives are defined as in Section 2.3. However, as already noticed, the notion of monotonicity depends on the considered ordering defined on $\mathcal{L}$. Here we then have to consider monotonicity with respect to $\preceq_L$.

With respect to the Pareto ordering $\preceq_p$, the standard negation $N((a,b)) = (b,a)$ is decreasing. However it is not for the lexicographic ordering $\preceq_L$ and is hence not a negation. Therefore, we propose a new definition of negation, illustrated in Figure 7.

Definition 11. *The natural negation $n_{\preceq_L}$ associated with the lexicographic ordering is defined as the operator that reverses the ordering of the elements of $\mathcal{L}$.*

This definition of $n_{\preceq_L}$ is actually a negation (involutive and decreasing). This result is derived from the fact that $\preceq_L$ is a total ordering on $\mathcal{L}$.

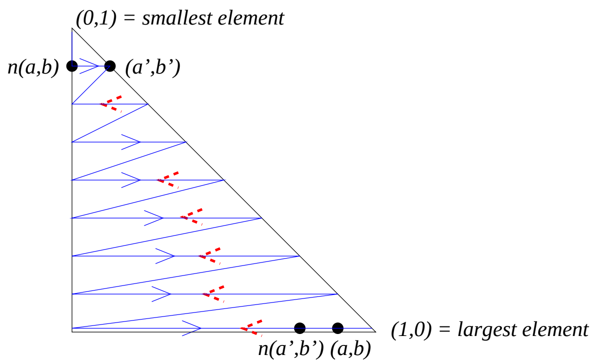


Figure 7: Natural negation for the lexicographic ordering. Plain arrows indicate the ordering from the smallest to the largest element of $\mathcal{L}$ and the dashed arrows indicate the reverse order. Two examples of points (a,b) and (a',b') and their negations $n_{\preceq_L}(a,b)$ and $n_{\preceq_L}(a',b')$ are shown.

From an algorithmical point of view, the computation of the negation is simple when the levels between 0 and 1 are discrete, i.e. take only a finite number of values (which is generally the case in practical applications). We tabulate the ranks of (a_i, b_j), for i and j varying from 0 to N if the interval $[0,1]$ is discretized on $N+1$ levels (for instance $a_i = \frac{i}{N}$, $b_j = \frac{j}{N}$). The rank of $(\frac{i}{N}, \frac{j}{N})$ is $r_{ij} = \frac{(N-j+1)(N-j)}{2} + i$ and the rank of $n_{\preceq_L}(\frac{i}{N}, \frac{j}{N})$ is equal to $\frac{(N+1)(N+2)}{2} - 1 - r_{ij}$.

From a geometrical point of view, the negation of a point (a,b) is the point $n_{\preceq_L}(a,b)$ such that the number of points in the triangle comprising the points smaller than (a,b) (see Figure 6) is equal to the number of points in the trapeze formed by the points that are larger than $n_{\preceq_L}(a,b)$.

Proposition 32. *The minimum* $\min_{\preceq_L}$ *and maximum* $\max_{\preceq_L}$ *associated with the lexicographic ordering are bipolar t-norms and t-conorms on the lattice* $(\mathcal{L}, \preceq_L)$[1]*. Moreover they are idempotent and mutually distributive,* $\min_{\preceq_L}$ *is the largest t-norm and* $\max_{\preceq_L}$ *the smallest t-conorm (according to* $\preceq_L$*). They are also dual with respect to the negation* $n_{\preceq_L}$*.*

Proof. It is easy to show that $\min_{\preceq_L}$ and $\max_{\preceq_L}$ are commutative, associative, increasing with respect to both arguments, and satisfy the boundary conditions of bipolar t-norms and t-conorms, directly from their definitions.

We have $\forall (a,b) \in \mathcal{L}, \min_{\preceq_L}((a,b),(a,b)) = (\min(a,a),b) = (a,b)$ hence $\min_{\preceq_L}$ is idempotent. Similarly $\max_{\preceq_L}$ is idempotent.

In order to show the distributivity property, let us consider any $(a,b),(a',b'),(a'',b'')$ in $\mathcal{L}$, with $(a',b') \preceq_L (a'',b'')$ (the case where the reverse inequality holds is similar). Then:

$$\max_{\preceq_L}((a,b), \min_{\preceq_L}((a',b'),(a'',b''))) = \max_{\preceq_L}((a,b),(a',b'))$$

and, from the increasingness of $\max_{\preceq_L}$:

$$\min_{\preceq_L}(\max_{\preceq_L}((a,b),(a',b')), \max_{\preceq_L}((a,b),(a'',b''))) = \max_{\preceq_L}((a,b),(a',b')).$$

Let C be a bipolar t-norm for $\preceq_L$ and $(a,b) \preceq_L (a',b')$. Then $\min_{\preceq_L}((a,b),(a',b')) = (a,b)$. From the increasingness of C we have: $C((a,b),(a',b')) \preceq_L C((a,b),(1,0))$ and since $C((a,b),(1,0)) = (a,b)$, it follows that $C((a,b),(a',b')) \preceq_L \min_{\preceq_L}((a,b),(a',b'))$. Hence $\min_{\preceq_L}$ is the largest bipolar t-norm for $\preceq_L$. The proof for bipolar t-conorms is similar.

The duality of $\min_{\preceq_L}$ and $\max_{\preceq_L}$ with respect to $n_{\preceq_L}$ is straightforward and directly follows from the fact that $n_{\preceq_L}$ reverses the order. □

Proposition 33. *The connective* I_N *defined as* $\forall (a,b) \in \mathcal{L}, \forall (a',b') \in \mathcal{L}$,

$$I_N((a,b),(a',b')) = \max_{\preceq_L}(n_{\preceq_L}(a,b),(a',b')) \tag{51}$$

is a bipolar implication.

Conversely, the negation can be deduced from the implication according to: $n_{\preceq_L}(a,b) = I_N((a,b),(0,1))$.

[1]Note that $\min_{\preceq_L}$ and $\max_{\preceq_L}$ are not increasing with respect to $\preceq_p$ and are therefore not t-norms and t-conorms on $(\mathcal{L}, \preceq_p)$.

Proof. This follows directly from Proposition 3. □

Proposition 34. *The connective I_R defined as $\forall(a,b) \in \mathcal{L}, \forall(a',b') \in \mathcal{L}$,*

$$I_R((a,b),(a',b')) = \bigvee_{\preceq_L}\{(\alpha,\beta) \in \mathcal{L} \mid \min_{\preceq_L}((a,b),(\alpha,\beta)) \preceq_L (a',b')\} \quad (52)$$

is a bipolar implication (it is the residual implication of the t-norm $\min_{\preceq_L}$). A closed-form expression is as follows:

$$I_R((a,b),(a',b')) = \begin{cases} (1,0) & \text{if } b > b' \\ (a',b') & \text{if } b < b' \\ (1,0) & \text{if } b = b' \text{ and } a \leq a' \\ (a',b') & \text{if } b = b' \text{ and } a > a' \end{cases} \quad (53)$$

or, equivalently:

$$I_R((a,b),(a',b')) = \begin{cases} (1,0) = 1_{\mathcal{L}} & \text{if } (a,b) \preceq_L (a',b') \\ (a',b') & \text{if } (a',b') \prec_L (a,b) \end{cases} \quad (54)$$

It is the adjoint of $\min_{\preceq_L}$, i.e.: $\forall(a_i,b_i) \in \mathcal{L}, i = 1...3$,

$$\min_{\preceq_L}((a_1,b_1),(a_2,b_2)) \preceq_L (a_3,b_3) \Leftrightarrow (a_2,b_2) \preceq_L I_R((a_1,b_1),(a_3,b_3)). \quad (55)$$

Proof. It is easy to show that I_R is decreasing in the first argument and increasing in the second one. Moreover, we have $I_R(0_{\mathcal{L}}, 0_{\mathcal{L}}) = 1_{\mathcal{L}}$, $I_R(0_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$, $I_R(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$, and $I_R(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$. Hence I_R is a bipolar implication.

Let us now show that $(I_R, \min_{\preceq_L})$ is an adjunction. Assume that

$$(a_2,b_2) \preceq_L I_R((a_1,b_1),(a_3,b_3)).$$

If $(a_1,b_1) \preceq_L (a_3,b_3)$, then this implies $I_R((a_1,b_1),(a_3,b_3)) = 1_{\mathcal{L}}$ and $(a_2,b_2) \preceq_L 1_{\mathcal{L}}$ which is always true. And $\min_{\preceq_L}((a_1,b_1),(a_2,b_2)) \preceq_L (a_1,b_1) \preceq_L (a_3,b_3)$. If $(a_3,b_3) \prec_L (a_1,b_1)$, then this implies $I_R((a_1,b_1),(a_3,b_3)) = (a_3,b_3)$ and $(a_2,b_2) \preceq_L (a_3,b_3)$. And $\min_{\preceq_L}((a_1,b_1),(a_2,b_2)) \preceq_L (a_2,b_2) \preceq_L (a_3,b_3)$. Therefore in both cases we have

$$(a_2,b_2) \preceq_L I_R((a_1,b_1),(a_3,b_3)) \Rightarrow \min_{\preceq_L}((a_1,b_1),(a_2,b_2)) \preceq_L (a_3,b_3).$$

The reverse implication can be proved in a similar way. □

This result is important for the construction of morphological operations, as will be seen next.

6.3 Algebraic and morphological dilations and erosions on the lattice $(\mathcal{B}, \preceq_L)$

Since $(\mathcal{B}, \preceq_L)$ is a complete lattice, algebraic dilations and erosions can be defined as in Section 3, as operators that commute with $\bigvee_{\preceq_L}$ and $\bigwedge_{\preceq_L}$, respectively. Similarly, the adjunction is defined with respect to $\preceq_L$.

The properties of these operators and their compositions (in particular closing and opening) are directly derived from the properties of complete lattices and are the same as those described in Section 3 for the general case.

Let us now consider the case where $\mathcal{S}$ is an affine space, on which translations are defined. Again, we define a degree of intersection as the supremum of a bipolar t-norm C and a degree of inclusion as the infimum of a bipolar implication I, where the bipolar connectives are defined according to $\preceq_L$.

Let (μ_B, ν_B) be a bipolar structuring element (in $\mathcal{B}$). The dilation and erosion of any element (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) are then expressed as:

$$\forall x \in \mathcal{S},\ \delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = {\bigvee_{\preceq_L}}_{y\in\mathcal{S}} C((\mu_B(x-y), \nu_B(x-y)), (\mu(y), \nu(y))). \tag{56}$$

$$\forall x \in \mathcal{S},\ \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = {\bigwedge_{\preceq_L}}_{y\in\mathcal{S}} I((\mu_B(y-x), \nu_B(y-x)), (\mu(y), \nu(y))). \tag{57}$$

In particular, we can use the lexicographic minimum $\min_{\preceq_L}$ as a t-norm. An example is illustrated in Figure 8.

As expected, the dilation extends the positive parts and reduces the negative parts. The priority given to the negative parts and the fact that $\min_{\preceq_L}$ always provides one of the input values (which is not the case of the Pareto ordering) induces a stronger effect of the transformation when using the lexicographic ordering (the Pareto minimum has the same negative part than $\min_{\preceq_L}$ and a smaller positive part).

It should be noted that, as in Section 4, a bipolar t-norm (i.e. a stronger operator than a general bipolar conjunction) is involved in the proposed definition (Equation 56), so as to guarantee good properties. For the erosion (Equation 57), both types of implications I_N and I_R can be used, with somewhat different properties.

Proposition 35. *The dilation defined from* $\min_{\preceq_L}$ *and the erosion defined from* I_N *(for* $\max_{\preceq_L}$ *and the negation* $n_{\preceq_L}$*) are dual with respect to the negation* $n_{\preceq_L}$*:* $\delta_{(\mu_B,\nu_B)}(n_{\preceq_L}(\mu,\nu)) = n_{\preceq_L}(\varepsilon_{(\mu_B,\nu_B)}(\mu,\nu))$.

Proof. Since I_N is an implication (Proposition 33), this follows from the duality of $\min_{\preceq_L}$ and $\max_{\preceq_L}$ with respect to $n_{\preceq_L}$ (Proposition 32). □

Proposition 36. *The dilation defined from* $\min_{\preceq_L}$ *and the erosion defined from* I_R *(residual implication of* $\min_{\preceq_L}$*) are adjoint. It follows that all general algebraic properties described in Sections 3 and 4 hold.*

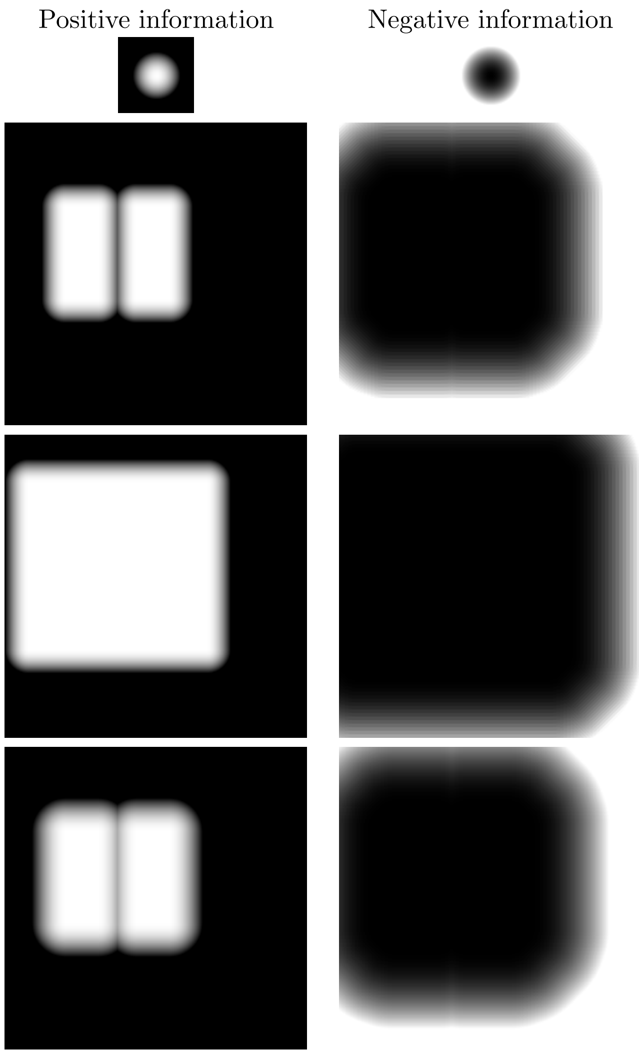


Figure 8: From top to bottom: bipolar fuzzy structuring element, original bipolar fuzzy set, dilation using the lexicographic minimum, dilation using Pareto ordering, for the sake of comparison. The grey levels encode the membership (or non-membership) values, ranking from 0 (black) to 1 (white).

Proof. This follows from the adjunction property of $\min_{\preceq_L}$ and I_R (Proposition 34). $\square$

Note that the two properties of adjunction and of duality are not simultaneously satisfied for these operators (since the dual operator of $\min_{\preceq_L}$ is $\max_{\preceq_L}$ but it is not its adjoint). It would be interesting to prove the existence and then build operators equivalent to Lukasiewicz ones, for $\preceq_L$, so as to derive results similar to those in the fuzzy case (see e.g. [17, 22, 30]) and in the bipolar fuzzy case for the Pareto ordering (see Section 5).

It follows that the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are true opening and closing if $\min_{\preceq_L}$ and I_R are used (because of the adjunction property), while they are not if $\min_{\preceq_L}$ and $\max_{\preceq_L}$ are used (they are not idempotent in this case).

Proposition 37. *Dilation and erosion defined by Equations 56 and 57 form an adjunction if and only if the involved C and I operators are adjoint. The general algebraic properties then hold (see Sections 3 and 4).*

Proposition 38. *The following properties hold:*

- *Equations 56 and 57 are consistent and provide results in $\mathcal{B}$.*
- *The dilation commutes with the supremum and the erosion with the infimum of the lattice $(\mathcal{B}, \preceq_L)$.*
- *Both operations are increasing with respect to $\preceq_L$.*
- *The dilation is extensive and the erosion is anti-extensive if and only if the origin of $\mathcal{B}$ completely belongs to the structuring element (i.e. $(\mu_B, \nu_B)(0) = 1_{\mathcal{L}}$).*
- *In the particular case where the set and the structuring element are not bipolar ($\nu = 1 - \mu$ and $\nu_B = 1 - \mu_B$), the definitions reduce to the classical ones in the fuzzy case.*
- *The following iterativity property holds:*

$$\delta_{(\mu_B,\nu_B)}(\delta_{(\mu'_B,\nu'_B)}((\mu,\nu))) = \delta_{\delta_{(\mu_B,\nu_B)}((\mu'_B,\nu'_B))}((\mu,\nu)). \tag{58}$$

Proof. The fact that dilation commutes with the supremum (and erosion with the infimum) is obvious in the case of adjunctions. In the more general case, it results from the fact that C distributes over the supremum and D over the infimum.

The increasingness is also straightforward in case of adjunctions. In the more general case, this is derived from the increasingness of the bipolar connectives and of the infimum and supremum with respect to $\preceq_L$.

Extensivity of dilation and anti-extensivity of erosion (iff $(\mu_B, \nu_B)(0) = 1_{\mathcal{L}}$) comes from the fact that $1_{\mathcal{L}}$ is the unit element of the bipolar t-norms, and $0_{\mathcal{L}}$ of the bipolar t-conorms.

The iterativity property directly follows from the associativity of the bipolar t-norm. $\square$

7 Derived operators

Once the two basic morphological operators, erosion and dilation, have been defined on bipolar fuzzy sets, a lot of other operators can be derived in a quite straightforward way. We provide a few examples in this section.

7.1 Morphological gradient

A direct application of erosion and dilation is the morphological gradient, which extracts boundaries of objects by computing the difference between dilation and erosion [88]. We propose here an extension to the bipolar fuzzy case.

Definition 12. *Let (μ, ν) a bipolar fuzzy set. We denote its dilation by a bipolar fuzzy structuring element by (δ^+, δ^-) and its erosion by $(\varepsilon^+, \varepsilon^-)$. We define the bipolar fuzzy gradient as:*

$$\nabla(\mu, \nu) = \bigwedge(N(\varepsilon^+, \varepsilon^-), (\delta^+, \delta^-)) \tag{59}$$

which is the set difference, expressed as the conjunction between (δ^+, δ^-) and the negation of $(\varepsilon^+, \varepsilon^-)$.

For instance, in the case of Pareto ordering and standard negation, the gradient is expressed as $\nabla(\mu, \nu) = (\min(\delta^+, \varepsilon^-), \max(\delta^-, \varepsilon^+))$.

Proposition 39. *The bipolar fuzzy gradient has the following properties:*

1. *Definition 12 defines a bipolar fuzzy set.*
2. *If the dilation and erosion are defined, in the case of Pareto ordering and using t-representable bipolar t-norms and t-conorms, we have:*

$$\nabla(\mu, \nu) = (\min(\delta_{\mu_B}(\mu), \delta_{\mu_B}(\nu)), \max(\varepsilon_{1-\nu_B}(\nu), \varepsilon_{1-\nu_B}(\mu))). \tag{60}$$

Moreover, if (μ, ν) is not bipolar (i.e. $\nu = 1 - \mu$), then the positive part of the gradient is equal to $\min(\delta_{\mu_B}(\mu), 1 - \varepsilon_{\mu_B}(\mu))$, which is exactly the morphological gradient in the fuzzy case.

Proof. These results follow directly from the expressions of bipolar dilations and erosions. □

An illustration is displayed in Figure 9. It illustrates both the imprecision (through the fuzziness of the gradient) and the indetermination (through the indetermination between the positive and the negative parts). The object is here somewhat complex, and exhibits two different parts, that can be considered as two connected components to some degree. The positive part of the gradient provides a good account of the boundaries of the union of the two components, which amounts to consider that the region between the two components, which has lower membership degrees, actually belongs to the object. The positive part has the expected interpretation as a surely possible position and spatial extension of the contours. The negative part shows the level of indetermination in the gradient: the gradient could be larger as well, and it could also include the region between the two components.

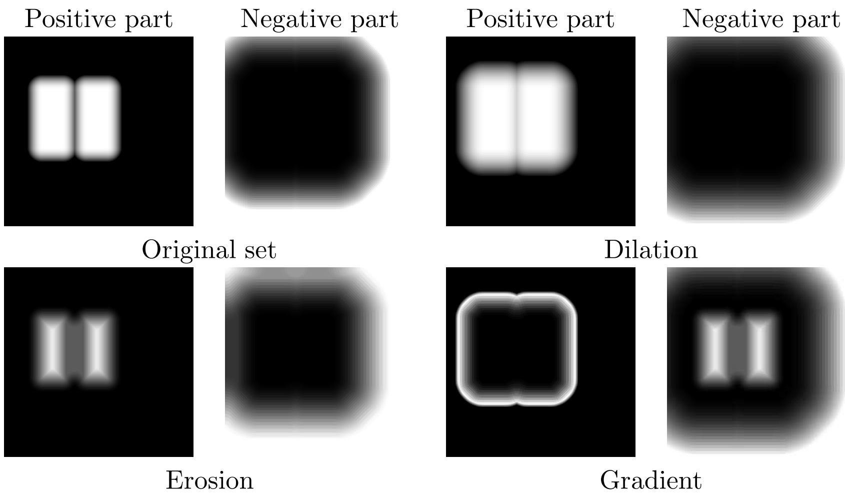


Figure 9: Bipolar morphological gradient using operators on the lattice $(\mathcal{B}, \preceq_p)$ and t-representable conjunction and implication derived from min and max. The structuring element is as in Figure 4.

The choice of the bipolar t-norms and t-conorms used for computing the dilation and the erosion have an influence on the result, with more or less effect, resulting from Proposition 25. In the case of Pareto ordering, finer results using C_W and D_W will be obtained than when using C_{min} and D_{max}, or C_{prod} and D_{sum} (see Proposition 27).

The choice of structuring element has also an influence. In the crisp continuous case, it can be shown that the difference between dilation and erosion tends towards the modulus of the gradient if the size of the structuring element tends towards 0 [88]. In the discrete case, it is then appropriate to use an elementary structuring element (according to the discrete connectivity defined on $\mathcal{S}$), i.e. the central point and its direct neighbors. In the fuzzy and bipolar cases, the structuring element can be somewhat more extended, in order to represent the local spatial imprecision and indetermination. This will lead to a larger gradient.

A direct application of Definition 12 is the computation of the perimeter of a bipolar fuzzy set, defined as a bipolar fuzzy number[2] $|\nabla(\mu, \nu)|$ where the

[2] A bipolar fuzzy number is a pair of fuzzy sets μ and ν such that μ and $1-\nu$ are fuzzy numbers and $\forall a \in \mathbb{R}\,(or\,\mathbb{N}), \mu(\alpha) + \nu(\alpha) \leq 1$. This definition can be relaxed by allowing $1-\nu$ be a fuzzy interval (i.e. its core is an interval). If both μ and $1-\nu$ are fuzzy intervals, then (μ, ν) will be called bipolar fuzzy interval.

cardinality $|.|$ is defined as proposed in [23, 25]:

Definition 13. *Let $(\mu, \nu) \in \mathcal{B}$. Its cardinality is defined as: $\forall n, |(\mu, \nu)|(n) = (|\mu|(n), 1 - |1 - \nu|(n))$.*

Proposition 40. *The cardinality introduced in Definition 13 is a bipolar fuzzy number on $\mathbb{N}$.*

In the spatial domain, the cardinality can be interpreted as the surface (in 2D) or the volume (in 3D) of the considered bipolar fuzzy set.

Definition 14. *Let (μ, ν) be a bipolar fuzzy set. Its perimeter (or surface) is defined as the bipolar fuzzy number $|\nabla(\mu, \nu)|$, where the gradient $\nabla(\mu, \nu)$ is given in Definition 12 and the cardinality $|.|$ in Definition 13.*

An example is shown in Figure 10.

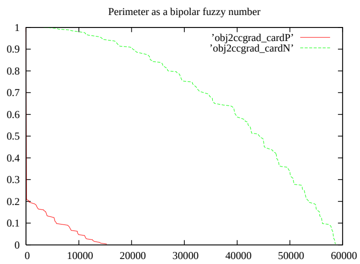


Figure 10: Perimeter of the bipolar fuzzy set shown in Figure 9 represented as a bipolar fuzzy number (the negative part is inverted), and computed as the cardinality of the gradient.

Other geometrical measures have been extended to the bipolar case in [23, 25].

7.2 Conditional operations and reconstruction

Another direct application of the basic operators concerns the notion of conditional dilation (respectively conditional erosion) [88]. These operations are very useful in mathematical morphology in order to constrain an operation to provide a result restricted to some region of space. In the digital case, a conditional dilation can be expressed using the intersection of the usual dilation with an elementary structuring element and the conditioning set. This operation is iterated in order to provide the conditional dilation with a larger structuring element. Iterating this operation until convergence leads to the notion of reconstruction. This operation is typically used in order to gain in robustness in cases we have a marker of some objects, and we want to recover the whole objects marked by this marker, and only these objects.

The extension of these types of operations to the bipolar fuzzy case is straightforward: given a bipolar fuzzy marker (μ_M, μ_N), the dilation of (μ_M, μ_N), conditionally to a bipolar fuzzy set (μ, ν) is simply defined as the conjunction of the dilation of (μ_M, μ_N) and (μ, ν).

Definition 15. *Let (μ, ν) a bipolar fuzzy set and (μ_M, ν_M) a bipolar fuzzy set considered as a marker. The conditional dilation is defined as:*

$$\delta((\mu_M, \nu_M)|(\mu, \nu)) = \bigwedge(\delta(\mu_M, \nu_M), (\mu, \nu)). \tag{61}$$

It is easy to show that this defines a bipolar fuzzy set.

In the case of Pareto ordering, this is expressed as: $(\min(\delta^+(\mu_M, \mu_M), \mu), \max(\delta^-(\mu_M, \nu_M), \nu))$, where δ^+ denotes the positive part of the dilation and δ^- its negative part. Since δ^+ can be interpreted as a fuzzy dilation and δ^- as a fuzzy erosion (see Section 5.3), the positive part of the conditional dilation corresponds to a fuzzy conditional dilation of μ (positive part of the initial bipolar fuzzy set), and its negative part corresponds to a fuzzy conditional erosion of ν.

Definition 16. *The reconstruction of a bipolar fuzzy set (μ, ν) according to the marker (μ_M, ν_M) is obtained from the iteration of conditional dilations until convergence:*

$$R((\mu, \nu), (\mu_B, \nu_B)) = [\delta((\mu_M, \nu_M)|(\mu, \nu))]^\infty. \tag{62}$$

This directly extends the corresponding classical notions in mathematical morphology [88].

An example is shown in Figure 11, showing that the conditional dilation of the marker is restricted to only one component (the one including the marker) of the original object (only the positive parts are shown). Iterating further this dilation would provide the whole marked component.

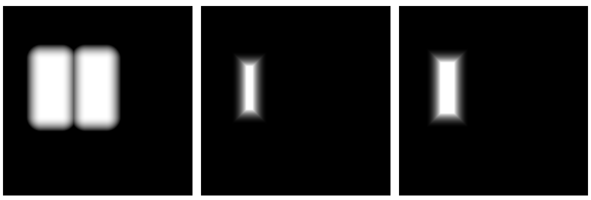


Figure 11: Conditioning set, marker and conditional dilation (only the positive parts are shown), on the lattice $(\mathcal{B}, \preceq_p)$.

Similar definitions can be given for conditional erosion (disjunction with the original bipolar fuzzy set) and reconstruction by erosion.

Note that, to be consistent with the geodesic framework, where the conditional dilation can be expressed according to the geodesic distance in the conditioninf set, in the digital case, dilations have to be performed with an elementary structuring element [88]. Here, a crisp non bipolar elementary element can be used as well, but it can be interesting to consider also the smallest bipolar fuzzy structuring element representing local imprecision and bipolarity. This can be further investigated for each specific application. By denoting (μ_B, ν_B) this elementary structuring element, the reconstruction is then computed according to the following sequence:

$$\begin{array}{rcl}
\delta^0 & = & \bigwedge((\mu_M, \nu_M), (\mu, \nu)) \\
\delta^1 & = & \bigwedge(\delta_{(\mu_B, \nu_B)}(\delta^0), (\mu, \nu)) \\
 & ... & \\
\delta^k & = & \bigwedge(\delta_{(\mu_B, \nu_B)}(\delta^{k-1}), (\mu, \nu)) \\
 & ... &
\end{array}$$

and the convergence is achieved for n such that $\delta^{n+1} = \delta^n$ (this occurs in a finite number of steps in a discrete bounded (finite) space).

7.3 Opening, closing, and derived operators

In a general algebraic setting, a filter on a complete lattice is defined as an idempotent and increasing operator. An opening γ is an anti-extensive filter and a closing φ is an extensive filter [85].

General properties of γ and φ hold in the lattice $(\mathcal{B}, \preceq)$, as in any complete lattice, whatever the choice of $\preceq$, thanks to the strong algebraic framework and the results of Section 3. In particular we have:

- typical examples of opening and closing are $\gamma = \delta\varepsilon$ and $\varphi = \varepsilon\delta$ where (ε, δ) is an adjunction;
- if (γ_i) is a family of openings, then $\gamma = \vee_i \gamma_i$ is an opening, and if (φ_i) is a family of closings, then $\varphi = \wedge_i \varphi_i$ is a closing;
- by denoting $Inv(\gamma)$ the invariant elements by γ (i.e. bipolar fuzzy sets (μ, ν) such that $\gamma((\mu, \nu)) = (\mu, \nu)$), an opening can be expressed as $\gamma((\mu, \nu)) = \vee\{(\mu', \nu') \in Inv(\gamma) \mid (\mu', \nu') \preceq (\mu, \nu)\}$ [61]. A similar expression holds for φ.

In practice, the morphological forms of erosions and dilations are often used to derive opening and closing. In $(\mathcal{B}, \preceq)$, we have the following monotony properties, for any dilation δ and erosion ε by the same bipolar fuzzy structuring element (omitted in the notations).

Proposition 41. *For any family of bipolar fuzzy sets (μ_i, ν_i), the following* *inequalities hold:*

$$\vee_i \delta\varepsilon(\mu_i, \nu_i) \preceq \delta\varepsilon(\vee_i(\mu_i, \nu_i)) \quad (63)$$
$$\vee_i \varepsilon\delta(\mu_i, \nu_i) \preceq \varepsilon\delta(\vee_i(\mu_i, \nu_i)) \quad (64)$$
$$\delta\varepsilon(\wedge_i(\mu_i, \nu_i)) \preceq \wedge_i \delta\varepsilon(\mu_i, \nu_i) \quad (65)$$
$$\varepsilon\delta(\wedge_i(\mu_i, \nu_i)) \preceq \wedge_i \varepsilon\delta(\mu_i, \nu_i) \quad (66)$$

Proof. These results directly follow from the fact that δ commutes with $\vee$, ε commutes with $\wedge$, and from Proposition 18. □

As an example, we consider the lattice $(\mathcal{B}, \preceq_p)$. The closing (obtained using Lukasiewicz operators) of the bipolar fuzzy object shown in Figure 9 is displayed in Figure 12. The small region between the two components in the positive part has been included in this positive part (to some degree) by the closing, which is the expected result.

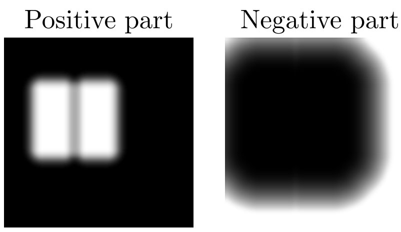


Figure 12: Bipolar fuzzy closing. The fuzzy bipolar structuring element (μ_B, ν_B) of Figure 4 was used here.

Another example is shown in Figure 13, where some small parts have been introduced in the bipolar fuzzy set (indicated by circles in the figure). The opening successfully removes these small parts (i.e. small regions with high μ values and small regions with low ν values are removed from the positive part and the negative part, respectively). A typical use of this operation is for situations where the initial bipolar fuzzy set represents possible/forbidden regions for an object. If we have some additional information on the size of the object, so that it is sure that it cannot fit into small parts, then opening can be used to remove possible small places, and to add to the negative part such small regions.

These operations have simpler expressions if the structuring element is not bipolar (i.e. $\nu_B = 1 - \mu_B$). The positive part of the opening is then the fuzzy opening, using Lukasiewicz operators, of μ by μ_B and its negative part is the fuzzy closing of ν by μ_B. Similar equivalences hold for closing.

From these new operators, a lot of other ones can be derived, extending the classical ones to the bipolar case. For instance, several filters can be deduced from opening and closing, such as alternate sequential filters [88], by applying alternatively opening and closing, with structuring elements of increasing size.

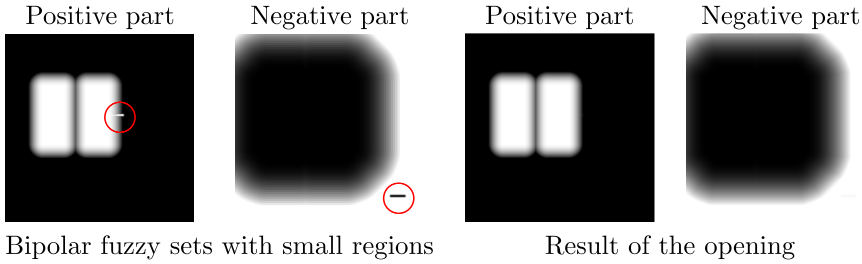


Figure 13: Bipolar fuzzy opening. Circles indicate small regions that are removed by the opening (see text). The bipolar fuzzy structuring element (μ_B, ν_B) of Figure 4 was used in this example.

Another example is the top-hat transform [88], which allows extracting bright structures having a given approximative shape, using the difference between the original image and the result of an opening using this shape as a structuring element. Such operators can be directly extended to the bipolar case using the proposed framework.

7.4 Distance from a point to a bipolar fuzzy set

While there is a lot of work on distances and similarity between interval-valued fuzzy sets or between intuitionistic fuzzy sets (see e.g. [93, 97]), none of the existing definitions addresses the question of the distance from a point to a bipolar fuzzy set, nor includes the spatial distance in the proposed definitions, although this is very useful for handling spatial information and for spatial reasoning. As in the fuzzy case [14], we propose to define the distance from a point to a bipolar fuzzy set using a morphological approach. In the crisp case, the distance from a point x to a set X is equal to n iff x belongs to the dilation of size n of X (the dilation of size 0 being the identity), but not to dilations of smaller size (it is sufficient to test this condition for $n-1$ in the discrete case). The transposition of this property to the bipolar fuzzy case leads to the following novel definition, using bipolar fuzzy dilations introduced in [23].

Definition 17. *The distance from a point x of $\mathcal{S}$ to a bipolar fuzzy set (μ, ν) ($\in \mathcal{B}$) is defined as:*

$$d(x, (\mu, \nu))(0) = (\mu(x), \nu(x)), \tag{67}$$

and

$$\forall n \in \mathbb{N}^*, d(x, (\mu, \nu))(n) = \delta^n_{(\mu_B, \nu_B)}(x) \wedge N(\delta^{n-1}_{(\mu_B, \nu_B)}(x)), \tag{68}$$

where N is a complementation (typically the standard negation $N(a, b) = (b, a)$ when Pareto ordering is used, or $n_{\preceq_L}$ for lexicographic ordering) and $\delta^n_{(\mu_B, \nu_B)}$ denotes n iterations of the dilation, using the bipolar fuzzy set (μ_B, ν_B) as structuring element.

In order to clarify the meaning of this definition, let us consider operations on the lattice $(\mathcal{B}, \preceq_p)$ and the case where the structuring element is not bipolar, i.e. $\nu_B = 1 - \mu_B$. Then the dilation writes: $\delta_{(\mu_B, 1-\mu_B)}(\mu, \nu) = (\delta_{\mu_B}(\mu), \varepsilon_{\mu_B}(\nu))$, where $\delta_{\mu_B}(\mu)$ is the fuzzy dilation of μ by μ_B and $\varepsilon_{\mu_B}(\nu)$ is the fuzzy erosion of ν by μ_B (see [30] for fuzzy mathematical morphology). The bipolar degree to which the distance from x to (μ, ν) is equal to n then writes: $d(x, (\mu, \nu))(n) = (\delta^n_{\mu_B}(\mu) \wedge \varepsilon^{n-1}_{\mu_B}(\nu), \varepsilon^n_{\mu_B}(\nu) \vee \delta^{n-1}_{\mu_B}(\mu))$, i.e. the positive part is the conjunction of the positive part of the dilation of size n (i.e. a dilation of the positive part of the bipolar fuzzy object) and the negative part of the dilation of size $n-1$ (i.e. an erosion of the negative part of the bipolar fuzzy object), and the negative part is the disjunction of the negative part of the dilation of size n (erosion of ν) and the positive part of the dilation of size $n-1$ (dilation of μ).

Proposition 42. *The distance introduced in Definition 17 has the following properties:*

- *it is a bipolar fuzzy set on* $\mathbb{N}$*;*
- *it reduces to the distance from a point to a fuzzy set, as defined in [14], if* (μ, ν) *and* (μ_B, ν_B) *are not bipolar (hence the consistency with the classical definition of the distance from a point to a set is achieved as well);*
- *the distance is strictly equal to 0 (i.e.* $d(x, (\mu, \nu))(0) = 1_{\mathcal{L}}$ *and* $\forall n \neq 0, d(x, (\mu, \nu))(n) = 0_{\mathcal{L}}$*) iff* $(\mu, \nu)(x) = 1_{\mathcal{L}}$*, i.e.* x *completely belongs to the bipolar fuzzy set.*

An example is shown in Figure 14. The results are in agreement with what would be intuitively expected. The positive part of the bipolar fuzzy number is put towards higher values of distances when the point is moved to the right of the object. After a number n of dilations, the point completely belongs to the dilated object, and the value to which the distance is equal to n', with $n' > n$, becomes $0_{\mathcal{L}} = (0, 1)$. Note that the indetermination in the membership or non-membership to the object (which is truly bipolar in this example) is also reflected in the distances.

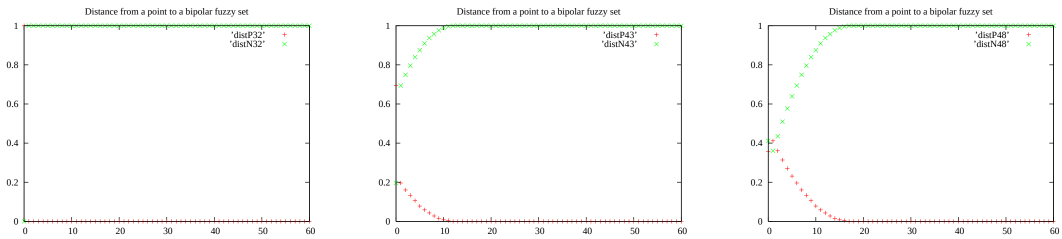
These distances can be easily compared using the extension principle[3], providing a bipolar degree $d_{\leq}$ to which a distance is less than another one. For the examples in Figure 14, we obtain for instance : $d_{\leq}[d(x_1, (\mu, \nu)) \leq d(x_2, (\mu, \nu))] = [0.69, 0.20]$ where x_i denotes the ith point from left to right in the figure. In this case, since x_1 completely belongs to (μ, ν), the degree to which its distance is less than the distance from x_2 to (μ, ν) is equal to $[\sup_a d^+(a), \inf_a d^-(a)]$, where d^+ and d^- denote the positive and negative parts of $d(x_2, (\mu, \nu))$. As another example, we have $d_{\leq}[d(x_5, (\mu, \nu)) \leq d(x_2, (\mu, \nu))] = [0.03, 0.85]$, reflecting that x_5 is clearly not closer to the bipolar fuzzy set (μ, ν) than x_2.

[3]An equivalent of the extension principle writes [63, 98] $((\mu_1, \nu_1) \circledast (\mu_2, \nu_2))(\gamma) = \vee_{\gamma = \alpha \circledast \beta} (\mu_1, \nu_1)(\alpha) \wedge (\mu_2, \nu_2)(\beta)$, where $\circledast$ denotes any operation. This principle can in particular be applied to define operations on bipolar fuzzy numbers or intervals.

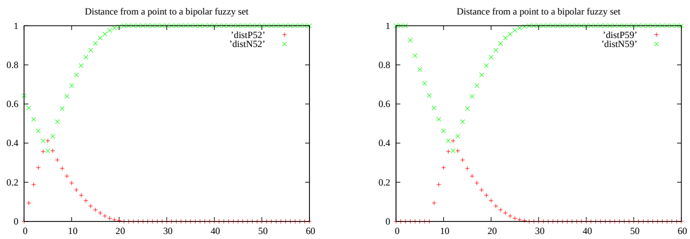
Bipolar fuzzy object:

positive part negative part

Test points in red (numbered 1..5 from left to right)



point 1 point 2 point 3



point 4 point 5

Figure 14: A bipolar fuzzy set and the distances from 5 different points to it, represented as bipolar fuzzy numbers (the positive part is shown in red and the negative part in green).

8 Conclusion

In this paper, we introduced a general algebraic framework for handling bipolar information using mathematical morphology operators. The case of bipolar fuzzy sets has been detailed, since it is general enough to cover several other settings. The general setting applies for any partial ordering inducing a complete lattice, and we have shown that strong properties can be derived in the general case. Two particular orderings have been detailed: Pareto ordering and lexicographic ordering. Other ones could be considered as well, and their choice should depend on their properties and their adequation with the domain of application and the associated semantics.

Examples on the potential use of the new reasoning tools provided by morphological operations have been sketched for both spatial reasoning and preferences modeling. Developing further these applications, along with a deeper investigation of derived operators, with appropriate choices of partial ordering, is the aim of our future work. Extensions to semi-lattices or general posets could be interestingly considered as well.

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