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Maximum Size of a Minimum Watching System and the Graphs Achieving the Bound

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Résumé

Soit $G = (V(G), E(G))$ un graphe non orienté. Un contrôleur w de G est une paire $w = (\ell(w), A(w))$, où $\ell(w)$ appartient à $V(G)$ et $A(w)$ est un ensemble de sommets de G à distance 0 ou 1 de $\ell(w)$. Si un sommet v appartient à $A(w)$, on dit que v est couvert par w. Deux sommets v_1 et v_2 de G sont dits séparés par un ensemble de contrôleurs si la liste des contrôleurs couvrant v_1 est différente de celle de v_2. On dit qu'un ensemble W de contrôleurs est un système de contrôle pour G si tout sommet v est couvert par au moins un contrôleur de W, et si deux sommets quelconques v_1, v_2 sont séparés par W. On dénote le nombre minimum de contrôleurs nécessaires pour contrôler G par $w(G)$. Nous donnons une borne supérieure sur $w(G)$ valable pour tout graphe connexe d'ordre n et nous caractérisons les arbres qui atteignent cette borne, avant d'étudier la caractérisation, plus compliquée, des graphes connexes qui l'atteignent.

Mots-clés : Théorie des graphes, Systèmes de contrôle, Codes identifiants

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Abstract Let $G = (V(G), E(G))$ be an undirected graph. A watcher w of G is a couple $w = (\ell(w), A(w))$, where $\ell(w)$ belongs to $V(G)$ and $A(w)$ is a set of vertices of G at distance 0 or 1 from $\ell(w)$. If a vertex v belongs to $A(w)$, we say that v is covered by w. Two vertices v_1 and v_2 in G are said to be separated by a set of watchers if the list of the watchers covering v_1 is different from that of v_2. We say that a set W of watchers is a watching system for G if every vertex v is covered by at least one $w \in W$, and any two vertices v_1, v_2 are separated by W. The minimum number of watchers necessary to watch G is denoted by $w(G)$. We give an upper bound on $w(G)$ for connected graphs of order n and characterize the trees achieving this bound, before studying the more complicated characterization of the connected graphs achieving this bound.

Key-words: Graph Theory, Watching Systems, Identifying Codes

1 Introduction

Let $G = (V(G), E(G))$ be an undirected connected graph (the case of an unconnected graph can also be treated, by considering separately its connected components). A *watcher* w of G is a couple $w = (\ell(w), A(w))$, where $\ell(w)$ belongs to $V(G)$ and $A(w)$ is a set of vertices of G at distance 0 or 1 from $\ell(w)$; in other words, $A(w)$ is a subset of $B(\ell(w))$, the ball of radius 1 centred at $\ell(w)$. We will say that w is *located* at $\ell(w)$ and that $A(w)$ is its *watching area* or *watching zone*. If a vertex v belongs to $A(w)$, we say that v is *covered* by w.

Two vertices v_1 and v_2 in G are said to be *separated* by a set of watchers if the list of the watchers covering v_1 is different from that of v_2.

We say that G is *watched* by a set W of watchers, or that W is a *watching system* for G if:

- for every v in $V(G)$, there exists $w \in W$ such that v is covered by w;
- if v_1 and v_2 are two vertices of G, v_1 and v_2 are separated by W.

Note that several watchers can be located at a same vertex, and a watcher does not necessarily cover the vertex where it is located.

The minimum number of watchers necessary to watch a graph G is denoted by $w(G)$.

We will often represent watchers simply by integers, as for the graph G_0 represented in Figure 1: the location of a watcher is written inside a rectangle; for each vertex v of the graph, the list of watchers covering v is written in italics, so that the watching area of each watcher can be retrieved. In the example of Figure 1, the watcher 1 is located at c and covers the vertices a, c and d, the watcher 2 is also located at c and covers the vertices b, c and e, the watcher 3 is located at f and covers the vertices d, e, f and h,

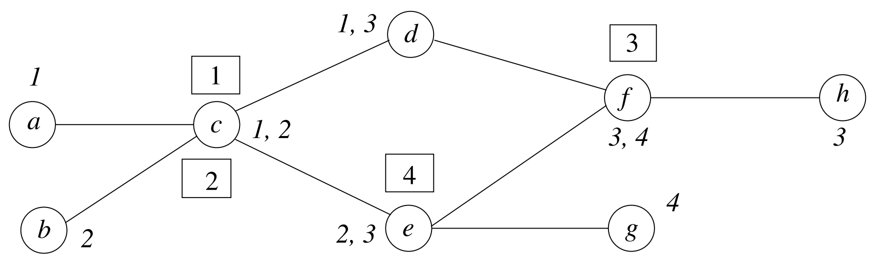


Figure 1: a graph G_0 watched by the watchers 1, 2, 3 and 4

and the watcher 4 is located at e and covers the vertices f and g. The graph G_0 is watched by these four watchers and it is easy to verify that $w(G_0) = 4$.

Let G be a graph of order n. If we have a set W of k watchers, the number of distinct non empty subsets of $\{\ell(w) : w \in W\}$ is equal to $2^k - 1$. Therefore, it is necessary to have $2^k - 1 \geq n$, and so we have the inequality:

$$w(G) \geq \lceil \log_2(n + 1) \rceil. \tag{1}$$

Obviously, watching systems generalize *identifying codes* (see the seminal paper [6], and [7] for a large bibliography): indeed, identifying codes are such that for any $w = (\ell(w), A(w)) \in W$, we have

$$A(w) = B(\ell(w)),$$

which means that, in this case, a watcher, or *codeword*, necessarily covers itself and all its neighbours.

See also [5], [8] for similar ideas.

Watching systems were introduced in [1], where motivations are exposed at large, basic properties are given, a complexity result is established, and the case of the paths is studied in detail, with comparison to identifying codes; see also [2].

In Section 2, we give an upper bound on $w(G)$ when G is a connected graph with n vertices. In Section 3, we characterize the trees of order n which achieve this bound: Theorems 7, 12 and 13 are for the cases $n = 3k$, $n = 3k + 2$ and $n = 3k + 1$, respectively. This helps to study, in Section 4, the characterization of *maximal* graphs reaching the bound, that is, graphs to which no edge can be added without decreasing the minimum number of necessary watchers: Theorems 15 and 16 give the answer for $n = 3k$ and $n = 3k + 2$ respectively, and Conjecture 17 is stated for the case $n = 3k + 1$. This in turn gives results for all the connected graphs achieving the bound.

2 The maximum minimum number of watchers

The following three easy lemmata will prove efficient. We recall that $H = (V(H), E(H))$ is a partial graph of $G = (V(G), E(G))$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

Lemma 1 *Let G be a graph and H be a partial graph of G. Then*

$$w(H) \geq w(G).$$

Proof. If H is watched by a set W of watchers, the same set W watches G, since two adjacent vertices in H are also adjacent in G. □

Note that this monotony property does not hold in general for identifying codes.

Lemma 2 *Let T be a tree, x be a leaf of T, and y be the neighbour of x.*

(a) *There exists a minimum watching system for T with one watcher located at y.*

(b) *If y has degree 2, there exists a minimum watching system for T with one watcher located at z, the second neighbour of y.*

Proof. (a) A watching system must cover x, so there is a watcher w_1 located at x or y, with $x \in A(w_1)$. If $w_1 = (x, A(w_1))$, then we can replace it by $w_2 = (y, A(w_1))$, since $B_1(y) \supseteq B_1(x)$.

(b) If $y \notin A(w_1)$, then one other watcher must cover y, and if $y \in A(w_1)$, then one must separate x and y, since $x \in A(w_1)$. In both cases, the task can be done by a watcher located at z. □

Lemma 3 *Let T be a tree of order 4 and let v be a vertex of T; there exists a set W of two watchers such that*

- *the vertices in $V(T) \setminus \{v\}$ are covered and pairwise separated by W — in this case, we shall say, with a slight abuse of notation, that $V(T) \setminus \{v\}$ is watched by W;*
- *the vertex v is covered by at least one watcher.*

Proof. On Figure 2, we give all possibilities: the two trees of order 4, and for each of them, the two locations for v (v is a leaf, or v is not a leaf). □

We are now ready to give an upper bound for $w(G)$ with respect to n, the order of G. Note in contrast that the upper bound for identifying codes, when they exist, is $n - 1$, see [3], [4], and is reached, among other graphs, by the star.

Theorem 4 *Let G be a connected graph of order n.*

- *If $n = 1$, $w(G) = 1$.*

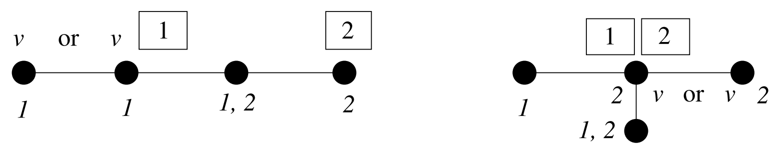


Figure 2: trees of order 4

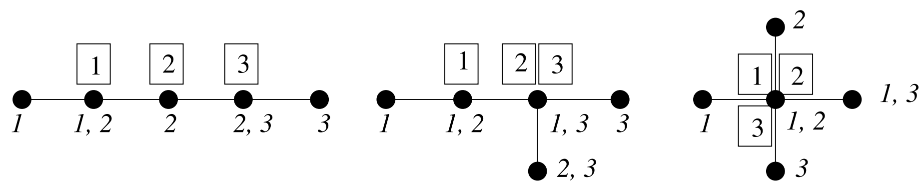


Figure 3: the case n=5 in Theorem 4

- *If $n = 2$ or $n = 3$, $w(G) = 2$.*
- *If $n = 4$ or $n = 5$, $w(G) = 3$.*
- *If $n \notin \{1, 2, 4\}$, $w(G) \leq \frac{2n}{3}$.*

The proof can be found in [1], [2], but we give it here, because the results of the four cases into which it is divided will be frequently used in the sequel.

Proof. For $n = 1$, $n = 2$, or $n = 3$, the result is direct. For $n = 4$, it is necessary to have at least $\lceil \log_2(5) \rceil = 3$ watchers and it is easy to verify that this is sufficient. For $n = 5$, all possibilities are given by Figure 3 and we can see that we always have $w(G) = 3$.

We proceed by induction on n. We assume that $n \geq 6$ and that the theorem is true for any connected graph of order less than n.

Let G be a connected graph of order n. Let T be a spanning tree of G; we will prove that $w(T) \leq \frac{2n}{3}$ and then the theorem will result from Lemma 1. We denote by D the diameter of T and we consider a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ of T, with length D.

We distinguish between four cases, according to some conditions on the degrees of v_{D-1} and v_{D-2}.

• *First case: the degree of v_{D-1} is equal to 3*
The vertex v_{D-1} is adjacent to a vertex x other than v_{D-2} and v_D; because D is the diameter, clearly x and v_D are leaves of T (see Figure 4). We consider the tree obtained by removing x, v_{D-1} and v_D from T; this new tree T' has order $n - 3$.

If $n \geq 8$ or if $n = 6$, we consider a minimum set W of watchers watching T'; if $n = 7$, then T' is of order 4, and, using Lemma 3, we choose a

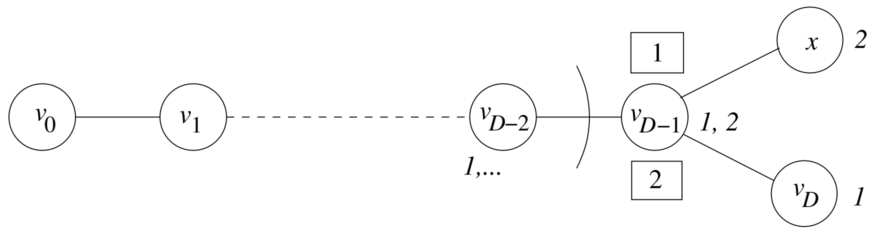


Figure 4: first case of Theorem 4: the degree of v_{D-1} is equal to 3

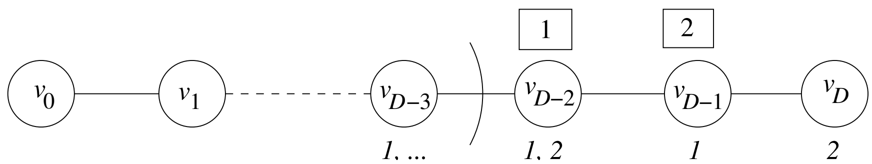


Figure 5: second case of Theorem 4: the degrees of v_{D-1} and v_{D-2} are equal to 2

set W of two watchers to watch $V(T') \setminus \{v_{D-2}\}$ *and* cover the vertex v_{D-2}.

Then for T, in both cases, we add to W two watchers $w_1 = (v_{D-1}, \{v_{D-2}, v_{D-1}, v_D\})$ and $w_2 = (v_{D-1}, \{v_{D-1}, x\})$. On Figure 4, we rename 1 and 2 these watchers. Then T is watched by $W \cup \{w_1, w_2\}$. So, $w(T) \leq |W| + 2 \leq w(T') + 2$.

Now we use the induction hypothesis: if $n \geq 8$ or $n = 6$, then $w(T) \leq \frac{2}{3}(n-3) + 2 = \frac{2n}{3}$; and if $n = 7$, then $w(T) \leq 2 + 2 = 4 < \frac{2}{3} \times 7$.

• *Second case: the degrees of v_{D-1} and v_{D-2} are equal to 2*

The neighbours of v_{D-1} are v_{D-2} and v_D, the neighbours of v_{D-2} are v_{D-3} and v_{D-1} (see Figure 5). We consider the tree obtained by removing v_{D-2}, v_{D-1} and v_D from T; this new tree T' has order $n-3$.

If $n \geq 8$ or if $n = 6$, we consider a minimum set W of watchers watching T'; if $n = 7$, T' is of order 4; again using Lemma 3, we choose a set W of two watchers to watch $V(T') \setminus \{v_{D-3}\}$ and cover the vertex v_{D-3}. As in the first case, we add to W two watchers: $w_1 = (v_{D-2}, \{v_{D-3}, v_{D-2}, v_{D-1}\})$ and $w_2 = (v_{D-1}, \{v_{D-2}, v_D\})$, and T is watched. So, $w(T) \leq |W| + 2 \leq w(T') + 2$. The end of this case is the same as in the first case.

• *Third case: the degree of v_{D-1} is at least 4*

The vertex v_{D-1} is adjacent to at least two vertices other than v_{D-2} and v_D: let x and y be two neighbours of v_{D-1} distinct from v_{D-2} and v_D; these two vertices are leaves of T (see Figure 6). We consider the tree T' obtained by removing x and y from T. By Lemma 2, there exists a minimum set W of

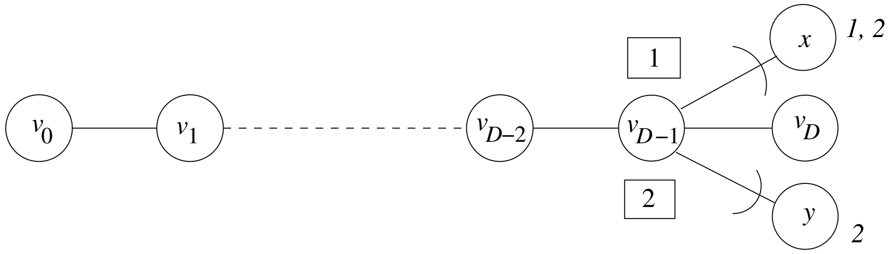


Figure 6: third case of Theorem 4: the degree of v_{D-1} is at least 4

watchers watching T' with a watcher w_1 located at v_{D-1}. For T, we take the set W and we add the watcher $w_2 = (v_{D-1}, \{x, y\})$; we also add the vertex x to the watching area of w_1. The tree T' being watched by W, the tree T is watched by $W \cup \{w_2\}$. So, $w(T) \leq w(T') + 1$.

If $n \geq 7$, the order of T' is at least 5 and, using the induction hypothesis, $w(T) \leq \frac{2}{3}(n-2) + 1 < \frac{2n}{3}$.

If $n = 6$, then $n - 2 = 4$ and $w(T) \leq 3 + 1 = 4 = \frac{2}{3} \times 6$.

• *Fourth case: the degree of v_{D-1} is equal to 2 and the degree of v_{D-2} is at least 3*

The neighbours of v_{D-1} are v_{D-2} and v_D. The vertex v_{D-2} is adjacent to v_{D-3} and v_{D-1} but also to at least one other vertex x (see Figure 7); if the degree of x is at least 3, using the fact that the diameter of T is equal to D, we can use the first or third case to conclude, with x playing the part of v_{D-1}.

So, we assume that the degree of x is 1 or 2; if its degree is 2, it has a neighbour y other than v_{D-2}.

We consider the tree T' of order $n-2$ obtained by removing v_{D-1} and v_D from T. By Lemma 2, there exists a minimum set W of watchers watching T' with a watcher w_1 located at v_{D-2}. To watch T, we take the set W and add the watcher $w_2 = (v_{D-1}, \{v_{D-1}, v_D\})$; we also add the vertex v_{D-1} to the watching area of w_1. Then T is watched by $W \cup \{w_2\}$.

The end of this case is exactly the same as in the previous case. □

Remark 5 *In the proof of Theorem 4, we have constructed, according to the cases, a tree T' with order $n-3$ such that $w(T) \leq w(T')+2$, or a tree T' with order $n-2$ such that $w(T) \leq w(T')+1$.*

These two constructions, from T to T', will be used several times in the sequel, e.g., in the proof of Theorem 7.

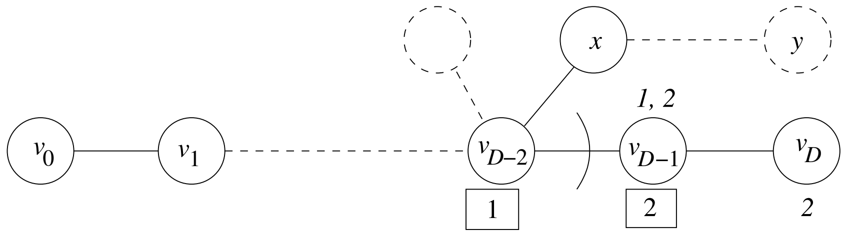


Figure 7: fourth case of Theorem 4: the degree of v_{D-1} is equal to 2 and the degree of v_{D-2} is at least 3

3 Trees T of order n achieving $w(T) = \lfloor \frac{2n}{3} \rfloor$

In this section, we characterize the trees T with n vertices and $w(T) = \lfloor \frac{2n}{3} \rfloor$. Our study is divided into three cases, $n = 3k$, $n = 3k + 2$ and $n = 3k + 1$.

We first define some particular trees, of order 1 to 5, that we name *gadgets*. For each gadget, we choose one or two particular vertex(ices) named *binding vertex(ices)*, through which the different gadgets will be exclusively connected between themselves; a vertex which is not a binding vertex is said to be *ordinary*. In the sequel, we will sometimes denote a gadget of order i by gi, $1 \le i \le 5$, and use the abbreviation b. v. for binding vertex. The gadgets are depicted in Figure 8; we represent the binding vertices with squares and ordinary vertices with circles.

We will use the following easy lemma, whose proof we omit.

Lemma 6 *Let T be a tree of order 3, and v and v' be two distinct vertices in T. It is possible to watch T with one watcher located at v and one watcher located at v'.*

As a consequence, if T' is a tree of order 4 and x is a leaf of T', there exists a set W of two watchers such that $T' \setminus \{x\}$ is watched by W and x is covered by W.

The following theorem characterizes the trees T with order $n = 3k$ and $w(T) = 2k$.

Theorem 7 *Let T be a tree of order $n = 3k$ for $k \ge 1$. We have:*
$w(T) = 2k \Leftrightarrow T$ can be obtained by choosing k gadgets of order 3 and joining these gadgets by their binding vertices to obtain a tree.

The tree T_{15} in Figure 9 is an example of a tree reaching this maximum.

Proof. Assume that a tree T of order $n = 3k$ is obtained by choosing k gadgets of order 3 and joining these gadgets by their b. v.'s to form a tree. It is clear that, to watch T, it is necessary to locate two watchers on each gadget. So T reaches the bound $2k$.

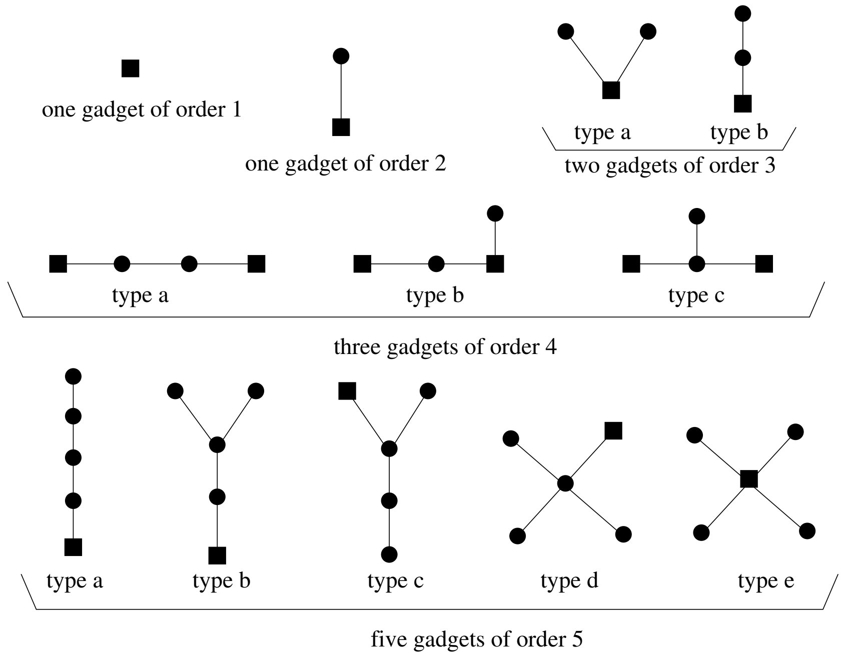


Figure 8: all the gadgets

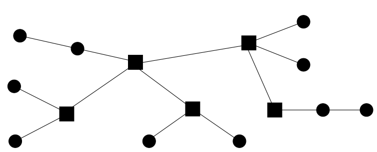


Figure 9: the tree T_{15}

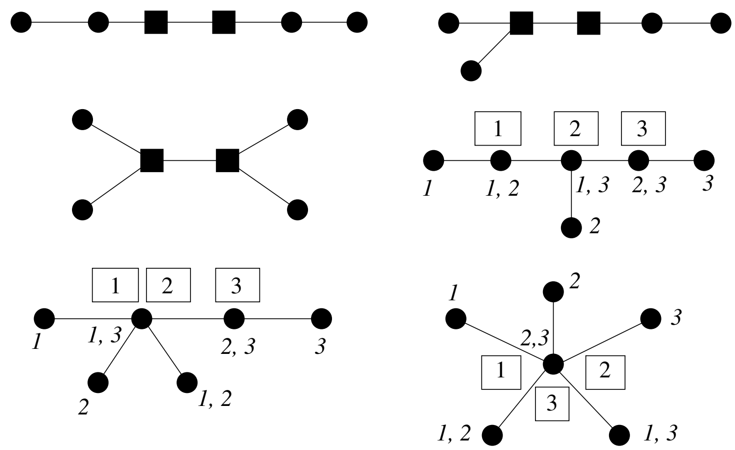


Figure 10: the trees of order 6 for the proof of Theorem 7

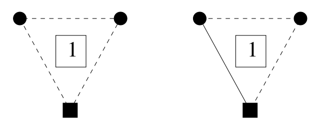


Figure 11: two representations for a g3 of type a or b

We will prove the converse by induction on k. For $k = 1$, it is immediate. We also examine the case $k = 2$, that is to say $n = 6$. We draw on Figure 10 the six different trees T on six vertices; when a tree is not of a type described in the right part of the equivalence, we explicitly give the watchers showing that $w(T) = 3$ and, in the other cases, we simply indicate the b. v.'s of the two gadgets involved.

We will sometimes represent a g3 of type a or b with a triangle, as on Figure 11: a dashed edge means that the edge may exist or not, with always exactly two edges in each g3. A watcher indicated inside the triangle means that this watcher is located at one of the three vertices of the triangle, at an appropriate vertex according to the case.

We assume now that $k \geq 3$ and that the theorem is true for $k' < k$. Let T be a tree of order $n = 3k$ with $w(T) = 2k$.

We consider again the proof of Theorem 4 using a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ of length D, where D is the diameter of T. Here, the third and fourth cases are impossible, because they imply that $w(T) < \frac{2n}{3} = 2k$, unless $n = 6$, which has just been dealt with. In the first case of Theorem 4, we rename by a, b, c and d respectively, the vertices v_{D-1}, v_D, x and v_{D-2}; in the second case, we rename by a, b, c and d respectively, the vertices v_{D-2}, v_{D-1}, v_D and v_{D-3}; in both cases, we remove the vertices a, b and c from T

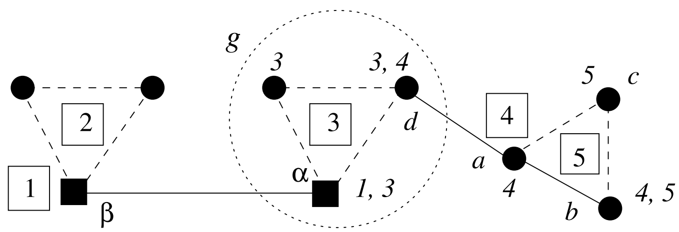


Figure 12: $2k-1$ watchers are sufficient in T (end of proof of Theorem 7)

and obtain a tree T' of order $3(k-1)$; by Remark 5, it appears that T' needs at least $w(T) - 2 = 2k - 2$ watchers and so $w(T') = 2(k-1)$ and we can apply the induction hypothesis to T': the vertex d belongs to a g3, g.

Assume that d is not the binding vertex of g. The b. v. α of g is adjacent to the b. v. β of another g3 in T' (cf. Figure 12). By Lemma 6, we can locate watchers w_4 and w_1 at a and β, so that d is covered by w_4 and α is covered by w_1; it is then possible to watch T with only one watcher located on the gadget g, as we can see on Figure 12, by choosing the appropriate vertex of g at which we locate the watcher denoted by 3. This leads to a contradiction on $w(T)$, and shows that d is the b. v. of g, in which case the result is immediately obtained, since $\{a, b, c\}$ can be seen as a g3, with its b. v. in a, connected to d. □

The following lemmata and definition will be used repeatedly in the sequel.

Lemma 8 *Let T be a tree of order 5 and v be a vertex of T. It is possible to watch T with three watchers, one of the three watchers being located at v.*

As a consequence, if T' is a tree of order 6 and x is a leaf of T', there exists a set W of three watchers such that $T' \setminus \{x\}$ is watched by W and x is covered by W.

Proof. The result for T is straightforward, by examining all the different possibilities, as we can see on Figure 13; the consequence on T' is immediate. □

Lemma 9 *Consider a g5 with binding vertex α and ordinary vertices v, x, y and z; there exists a set W of two watchers such that*

- *$\{x, y, z\}$ is watched by W;*
- *the vertex v is covered by W.*

Proof. If the g5 is of type a, b, c, or d, then the four vertices v, x, y, z form a tree, and by Lemma 3, we are done. If the g5 is of type e, then it is also possible, with two watchers located at α, the centre of the star, to watch $\{x, y, z\}$ and cover v. □

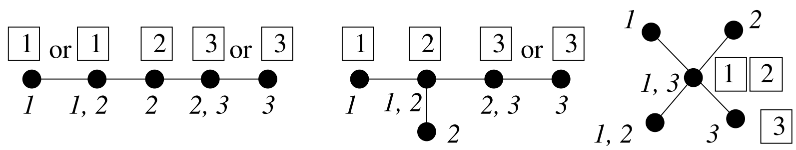
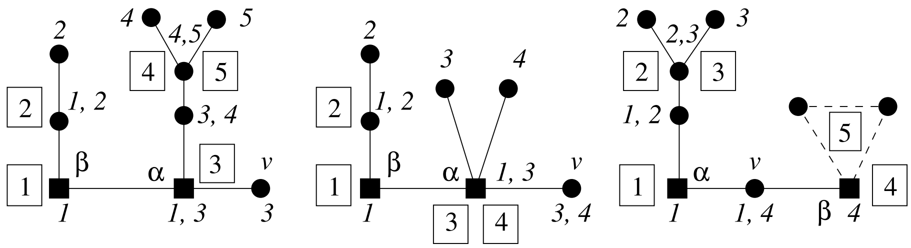


Figure 13: illustration for Lemma 8



(a) (b) (c)

Figure 14: illustration for Lemma 11

Definition 10 *Let $H = (V(H), E(H))$ be a connected graph and v be a vertex in $V(H)$; let H' be the graph obtained by removing the vertex v from H (H' is connected or not). We say that v is* free of charge*, or* free*, in H if there exists a minimum watching system for the graph H' which is also a watching system for H.*

Lemma 11 *Let p be an integer verifying $p \geq 2$. Let F be a forest obtained by choosing p gadgets of order 3 or 5 and possibly, if desired, by adding edges between the binding vertices of the p gadgets. Let v be a new vertex, which is adjacent to at least one binding vertex and cannot be adjacent to ordinary vertices; we assume that the graph obtained by adding v to F is a tree, T. Then, the vertex v is free in T.*

Proof. If v is adjacent to only one b. v., let α be this vertex; since T is connected and $p \geq 2$, the vertex α is adjacent to another b. v., β. If v is adjacent to at least two b. v.'s among the p gadgets, let α and β be two such vertices. Figure 14 illustrates the lemma in detail in three cases:

(a) v is linked to the b. v. α of a g5 and α is linked to the b. v. of a g3;
(b) v is linked to the b. v. α of a g3 and α is linked to the b. v. of a g3;
(c) v is linked to the b. v.'s of a g5 and of a g3.

The other cases, using repeatedly Lemmata 6 and 8, can be treated exactly in the same way. $\square$

We are now ready to characterize the trees T with order $n = 3k + 2$ and $w(T) = 2k + 1$.

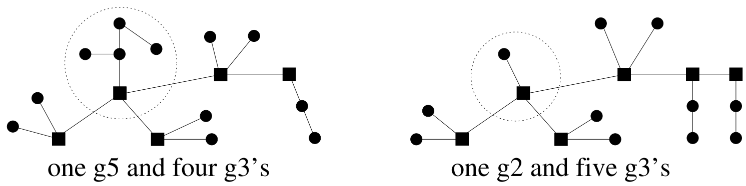


Figure 15: the trees T_{17} and T'_{17}

Theorem 12 *Let T be a tree of order $n = 3k+2$ for $k \geq 1$. We have: $w(T) = 2k+1 \Leftrightarrow T$ can be obtained by choosing one gadget of order 2 and k gadgets of order 3, or one gadget of order 5 and $k-1$ gadgets of order 3, and joining these gadgets by their binding vertices to obtain a tree.*

The trees T_{17} and T'_{17} of Figure 15 are examples of trees which achieve this maximum.

Proof. Assume that a tree T of order $n = 3k+2$ is obtained by choosing one g2 and k g3's, or one g5 and $k-1$ g3's, and finally joining these gadgets by their binding vertices, in order to obtain a tree. It is necessary to locate one watcher on a g2, two watchers on a g3 and, because a g5 has four ordinary vertices, three watchers on a g5. So T achieves the bound $2k+1$: if there is a g2, we need one watcher for the g2 and $2k$ watchers for the k g3's, if there is a g5, three watchers for the g5 and $2k-2$ watchers for the $k-1$ g3's.

We will prove the converse by induction on k. For $k = 1$, $n = 5$ and the result is clear, see Figure 3: T is a g5 (and in two out of three cases, it can also be seen as the connexion of a g2 and a g3). We assume now that $k \geq 2$ and that the theorem is true for $k' < k$. Let T be a tree of order $n = 3k+2$ with $w(T) = 2k+1$. We consider again the proof of Theorem 4, using a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ of length D, where D is the diameter of T.

• *Part (a): we assume that we are in the first or second case in the proof of Theorem 4*

In the first case, we rename by a, b, c and d respectively, the vertices v_{D-1}, v_D, x and v_{D-2}; in the second case, we rename by a, b, c and d respectively, the vertices v_{D-2}, v_{D-1}, v_D and v_{D-3}; we remove the vertices a, b and c from T and obtain a tree T' of order $3(k-1)+2$; by Remark 5, it appears that T' needs at least $w(T)-2 = 2k-1$ watchers and so $w(T') = 2(k-1)+1$ and we can apply the induction hypothesis to T': T' is of one of the two types described in the right part of the equivalence, and the vertex d belongs to a gadget g, whose b. v. we denote by α. Assume first that $d \neq \alpha$.

• (i) If g is of order 2, the subtree induced by the vertices of g and the vertices a, b and c yields a g5 of type a or b, and the result is proved for T.

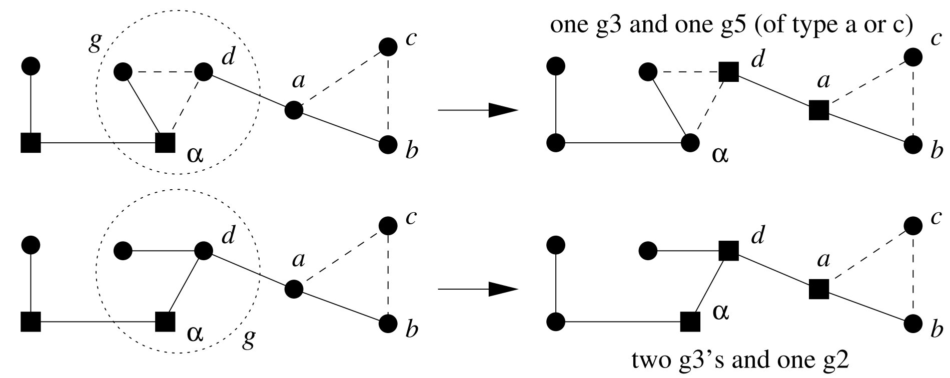


Figure 16: cases for $n = 8$ in part (a) of Theorem 12, when g is of order 3

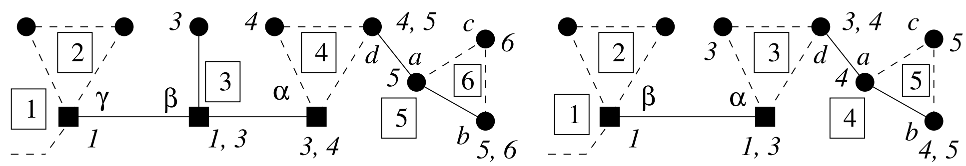


Figure 17: cases for $n \geq 11$ in part (a) of Theorem 12, when g is of order 3

• (ii) Assume next that g is of order 3. If T is of order 8, the two possibilities are given by Figure 16. If T is of order at least 11, there exists in T' a gadget g' connected to α, and we denote by β the b. v. of g'. If g' is a g2, which is itself connected to another g3 with binding vertex γ, then the left part of Figure 17 shows how to use only one watcher for g, which leads to a contradiction on $w(T)$; the same is true if g' is a g3, see the right part of Figure 17, which actually is the same as Figure 12; the case when g' is a g5 works similarly, using Lemma 8. So the only case left is when g' is a g2 connected only to g, and Figure 18 shows how to solve it.

• (iii) Finally, assume that g is of order 5. If T is of order 8, the reader will convince himself that locating d at all the different vertices, except at the b. v. α, of all the different types for a g5 leads to the six patterns given by Figure 19. If T is of order 11, the b. v. α is adjacent to the b. v. β of a g3. When one examines the different possibilities, it appears that if T reaches the bound, it is of the wished shape: this is shown by Figure 20.

To close the case when g is of order 5, we study the case when T is of order at least 14; then the tree T'' obtained from T' by removing the four vertices of g other than α has order at least 7 and we can apply Lemma 11 to it, which shows that the vertex α is free in T''. Using Lemma 9, we can use two watchers on g to watch $V(g) \setminus \{\alpha, d\}$ and cover the vertex d. With one

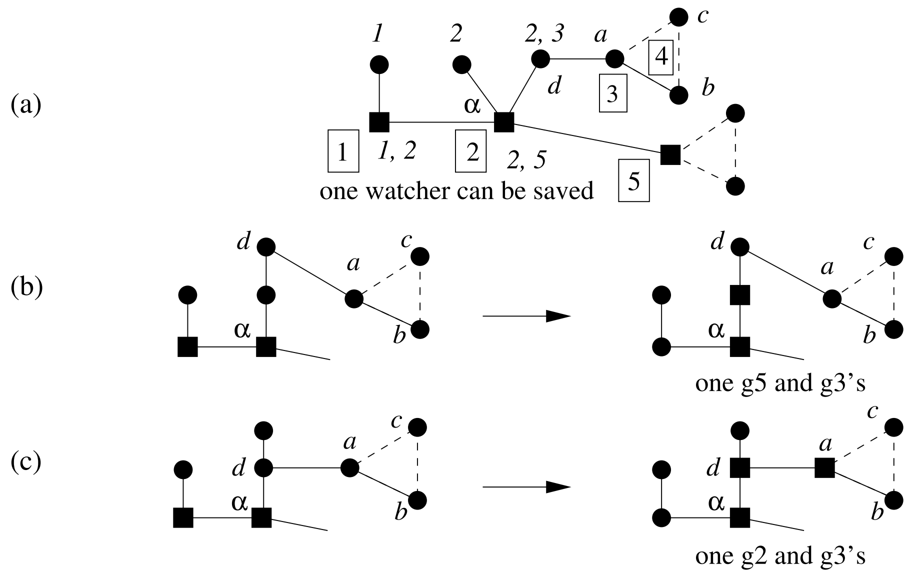


Figure 18: more cases for $n \geq 11$ in part (a) of Theorem 12, when g is of order 3

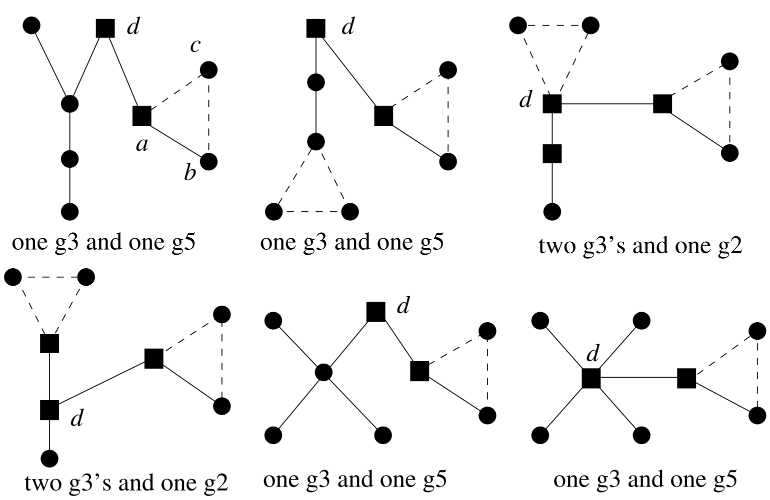


Figure 19: cases for $n = 8$ in part (a) of Theorem 12, when g is of order 5

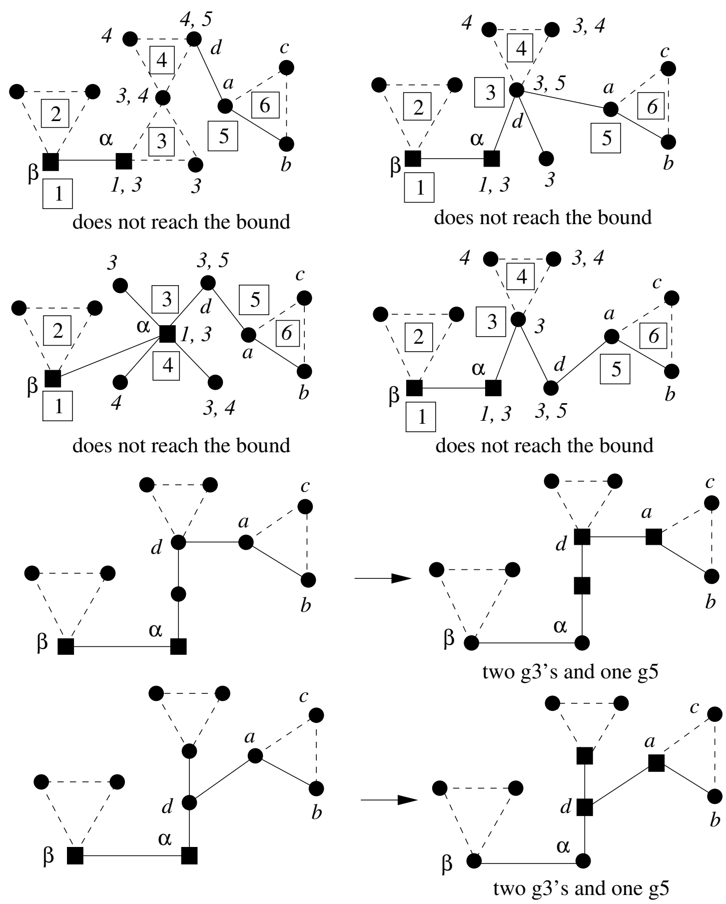


Figure 20: cases for $n = 11$ in part (a) of Theorem 12, when g is of order 5

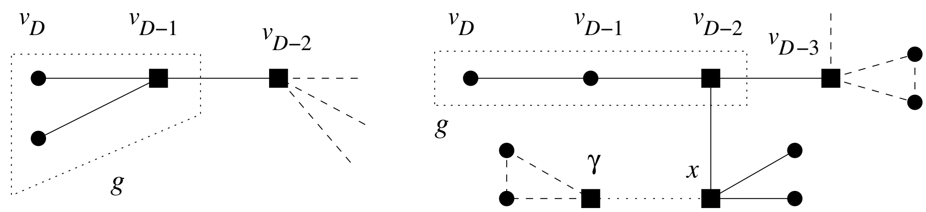


Figure 21: illustration for part (b) of Theorem 12

watcher at a covering d, we can separate d from all the other vertices: so, we can do with only two watchers on g, and T does not achieve the bound.

This shows that if $d \neq \alpha$, then either the tree does not achieve the bound, or it is of the desired form. On the other hand, if $d = \alpha$, then the result is immediately obtained. This ends part (a).

• *Part (b): we assume that we are in the third or fourth case in the proof of Theorem 4*

If we are in the third case, we remove the vertices x and y and if we are in the fourth case, we remove the vertices v_{D-1} and v_D; we obtain a tree T' of order $3k$. By Remark 5, we have $w(T') = 2k$ and Theorem 7 may be used: T' can be obtained as a collection of g3's linked by some edges between their binding vertices. So, the vertex v_0 is a leaf of a g3, g; now we reverse the longest path $v_0, v_1, \ldots, v_D$ in T. If g is of type a (see the left part of Figure 21), then v_{D-1} is linked to only one b. v., v_{D-2}, and has degree 3, because D is the diameter of the tree, and we are brought back to the first case. And if g is of type b, then v_{D-1} has degree 2, and either v_{D-2} has degree 2 and we are in the second case, or v_{D-2} has degree at least 3 and we are in the fourth case, with at least one b. v. x linked to v_{D-2} and x of degree at least 2 (see the right part of Figure 21); however, x cannot be linked to another b .v. γ, since this would increase the diameter of the tree, and for the same reason the g3 of x is of type a, so that necessarily x has degree 3. With x playing the part of v_{D-1}, we are again in the first case. In all cases, we can re-use the result obtained in part (a). $\square$

The last case, $n = 3k + 1$ and $w(T) = 2k$, offers the greatest number of possibilities for the gadgets.

Theorem 13 *Let T be a tree of order $n = 3k + 1$ for $k \geq 2$. We have: $w(T) = 2k \Leftrightarrow T$ can be obtained by choosing*

- (i) *two gadgets of order 2 and $k - 1$ gadgets of order 3,*
- (ii) *or one gadget of order 2, one gadget of order 5 and $k - 2$ gadgets of order 3,*

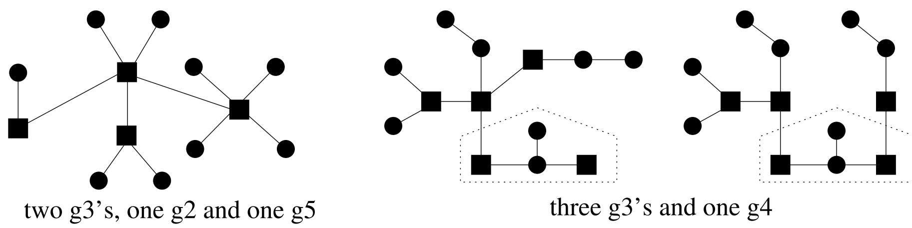


Figure 22: the trees T_{13}, T'_{13} and T''_{13}

- (iii) *or two gadgets of order 5 and* $k-3$ *gadgets of order 3,*
- (iv) *or one gadget of order 1 and* k *gadgets of order 3,*
- (v) *or one gadget of order 4 and* $k-1$ *gadgets of order 3,*

and joining these gadgets by their binding vertices to obtain a tree.

It may be that one of the two binding vertices of a g4 is not linked to any (binding) vertex. The trees T_{13}, T'_{13} and T''_{13} of Figure 22 are examples of trees achieving the bound $2k$ for $n = 3k+1$ (with $k = 4$).

Proof. Assume that a tree T of order $n = 3k+1$ is obtained as specified in the right part of the above equivalence. It is necessary to locate one watcher on a g2, two watchers on a g3 and two on a g4 (because a g4 has two ordinary vertices), and three watchers on a g5. So T reaches the bound $2k$:
if we are in (i), $(2 \times 1) + ((k-1) \times 2) = 2k$;
in (ii), $(1 \times 1) + (1 \times 3) + ((k-2) \times 2) = 2k$;
in (iii), $(2 \times 3) + ((k-3) \times 2) = 2k$;
in (iv), $(1 \times 0) + (k \times 2) = 2k$;
in (v), $(1 \times 2) + ((k-1) \times 2) = 2k$.
We will prove the converse by induction on k. For $n = 7$, the different possibilities are examined on Figure 23. Now, we assume that $n \geq 10$.

We use the same scheme of proof as for Theorem 12: we assume that $k \geq 3$ and that the theorem is true for $k' < k$, we let T be a tree of order $n = 3k+1$ with $w(T) = 2k$, and we consider the proof of Theorem 4, using a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ of length D, where D is the diameter of T.

• *Part (a): we assume that we are in the first or second case in the proof of Theorem 4*

If in T there is a g4, and if only one of its binding vertices is connected to another gadget, this g4 can be viewed as two g2's or as one g1 and one g3. So we can assume that, if there is a g4, i.e., if we are in case (v) of the theorem, then each of the two b. v.'s of the g4 is connected to at least one g3.

In the first case, we rename by a, b, c and d respectively, the vertices v_{D-1}, v_D, x and v_{D-2}; in the second case, we rename by a, b, c and d

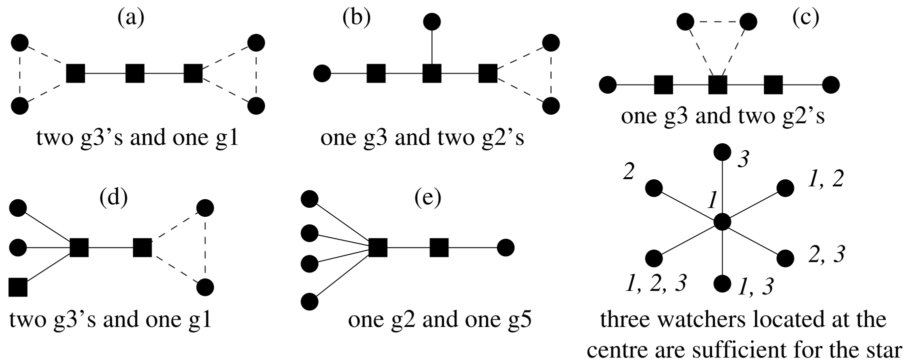


Figure 23: all the possibilities for $n = 7$ in Theorem 13

respectively, the vertices v_{D-2}, v_{D-1}, v_D and v_{D-3}. In both cases, we remove the vertices a, b and c from T and obtain a tree T' with order $3(k-1)+1$; by Remark 5, it appears that T' needs at least $w(T)-2 = 2k-2$ watchers and so $w(T') = 2(k-1)$: we can apply the induction hypothesis to T', which is of one of the five types described in the right part of the equivalence.

The vertex d belongs to a gadget g; as before, if d is a binding vertex, we are done, so we assume from now on that d is ordinary, so that g is of order 2 or more, and we have four cases, according to the order of g.

• (1) If g is of order 2, the subtree induced by the vertices of g and the vertices a, b and c yields a g5 and the result is proved: indeed, if T' has two g2's and $k-2$ g3's (case (i)), or one g2, one g5 and $k-3$ g3's (case (ii)), then T can be obtained with one g2, one g5 and $k-2$ g3's (case (ii)), or two g5's and $k-3$ g3's (case (iii)), respectively.

• (2) Assume next that g is of order 3, with binding vertex α.

If $n = 10$, we consider for T' the cases in Figure 23 where there is at least one g3, that is, the cases (a)–(d). The cases (a) and (d), where there is a g1, will be studied below, for general values of n. If, in the case (b), g is of type a, then, see Figure 24(a), five watchers are sufficient to watch T, whereas if g is of type b, then, according to the location of d in g, T consists of two g5's, or of one g2, one g3 and one g5, see Figure 24(b) and (c).

Similarly, if in the case (c) of Figure 23, g is of type a, then five watchers are sufficient, whereas if g is of type b, then T consists of one g2, one g3 and one g5, or of two g3's and two g2's — cf. Figure 30(a) below.

We consider now the case when there is a g1, with vertex δ, in T', with $n \geq 10$: we are in case (iv) of Theorem 13 and all the other gadgets in T' are g3's. If δ is linked neither to α nor to any neighbour of α, we are in the situation depicted by Figure 25(a) and $2k-1$ watchers are sufficient for T: since γ or δ is linked to another g3, these two vertices can be separated by another watcher. So we can assume from now on that δ is linked either to α

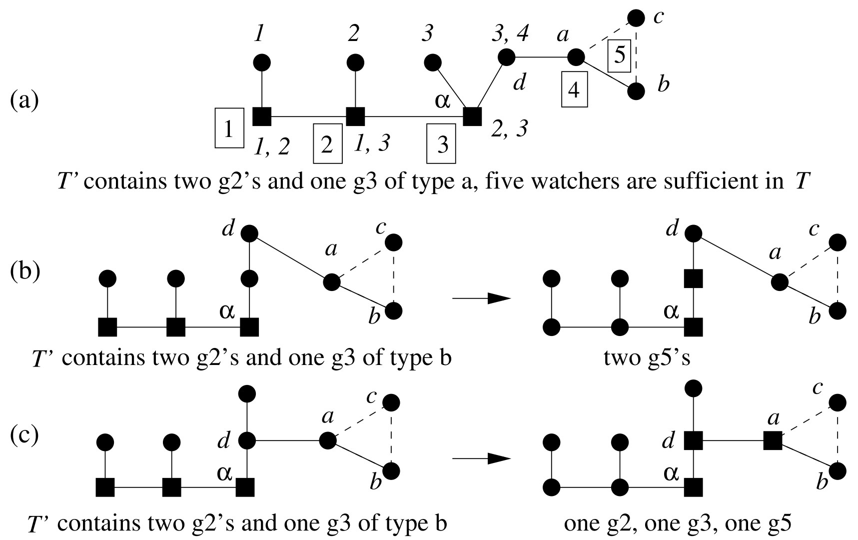


Figure 24: cases for $n = 10$ in part (a) of Theorem 13, when g is a g3

or to one of its neighbours. First, we assume that α is linked to at least one g3, cf. the left part of the tree in Figure 25(a). Again, we can save one watcher, so that T does not achieve the bound $2k$, unless we are in one of the following three cases:

(i) δ is linked only to α, see Figure 25(b). Then either δ is not covered by any watcher, or it is covered by the watcher 3 located at α, in which case no watcher separates e and δ. This gives the three possibilities detailed in Figure 26.

(ii) δ is linked to α and to exactly one other g3, which is not linked to any other g3, and the watcher 3 cannot be located at α, which means that g is of type b, see Figure 25(c), where γ and δ are not separated. Then there are two possibilities, given in Figure 27.

(iii) δ is linked to a neighbour β of α, and neither δ nor β is linked to other b. v.'s, and g is of type b, see Figure 25(d), where β and f are not separated. Figure 28 shows then that we still are in the conditions of Theorem 13.

Now we can assume that α is not linked to any g3, which means that it is linked to δ, which in turn is linked to at least one g3; then a g4 appears, containing g and δ, and with binding vertices d and δ. So the case when there is a g1 in T' is closed, also completing the case $n = 10$. From now on, we assume that $n \geq 13$ and that there is no g1 in T'.

We can remark the following: if in T' we have a g4, and if one of its binding vertices is connected to a g3, it is always possible to locate $w(T')$ watchers on T' with one watcher at the second binding vertex v of the g4, see Figure 29. Since we have required that each of the two b. v.'s of the g4

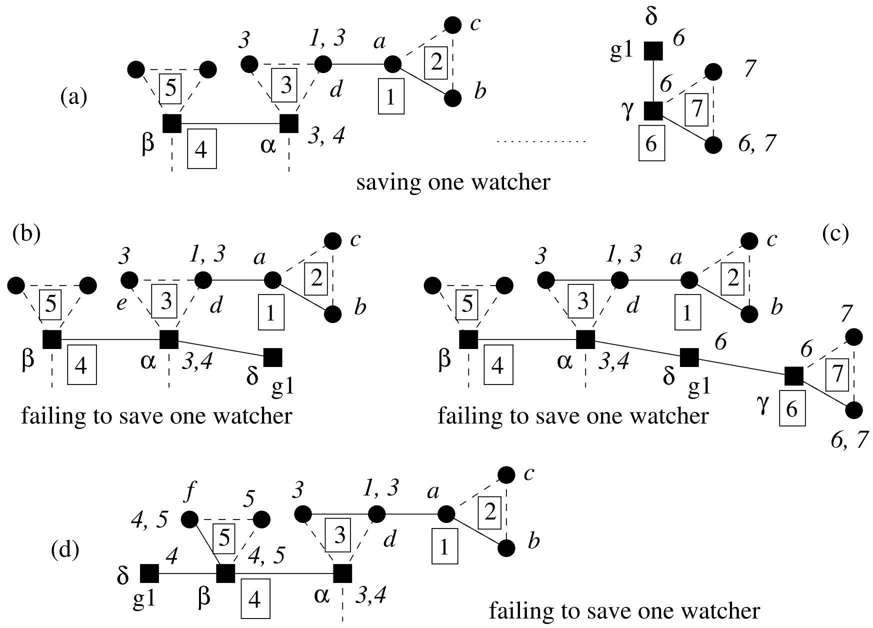


Figure 25: part (a) of Theorem 13: g is a g3 and there is a g1

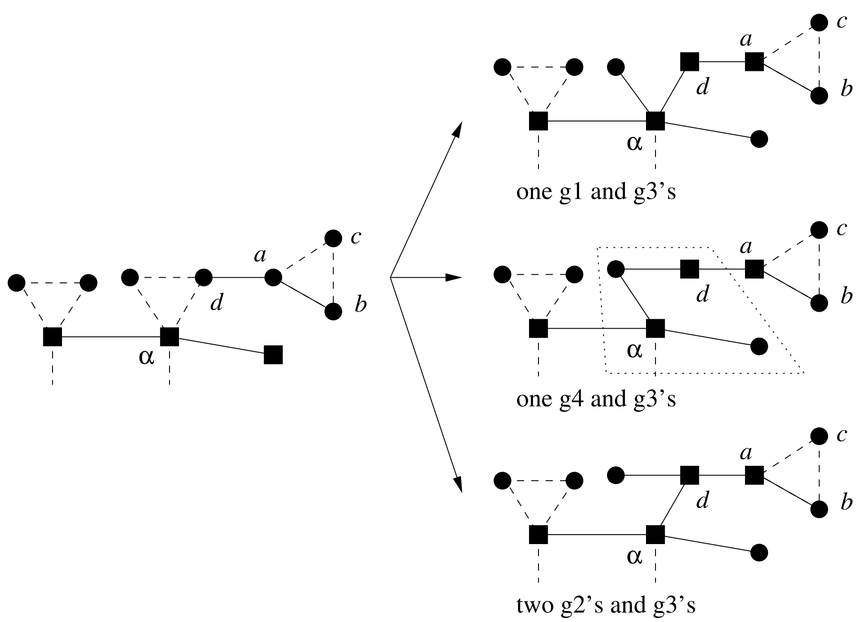


Figure 26: cases with a g1 in part (a) of Theorem 13, when g is a g3

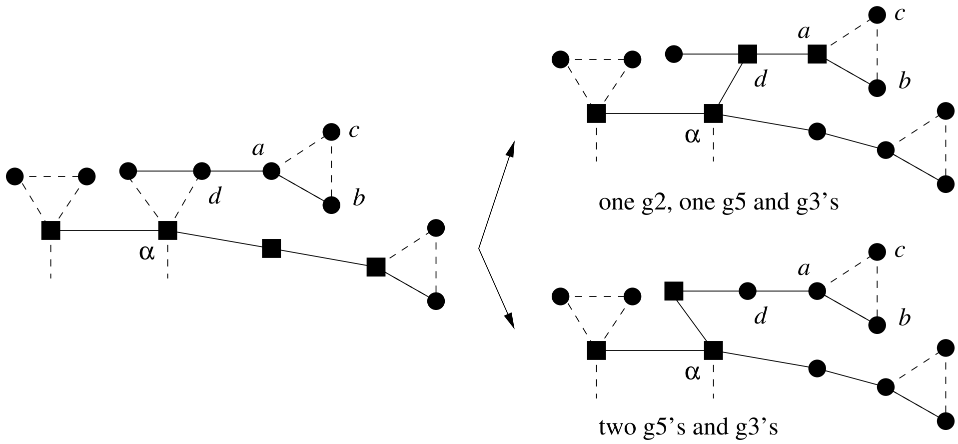


Figure 27: more cases with a g1 in part (a) of Theorem 13, when g is a g3

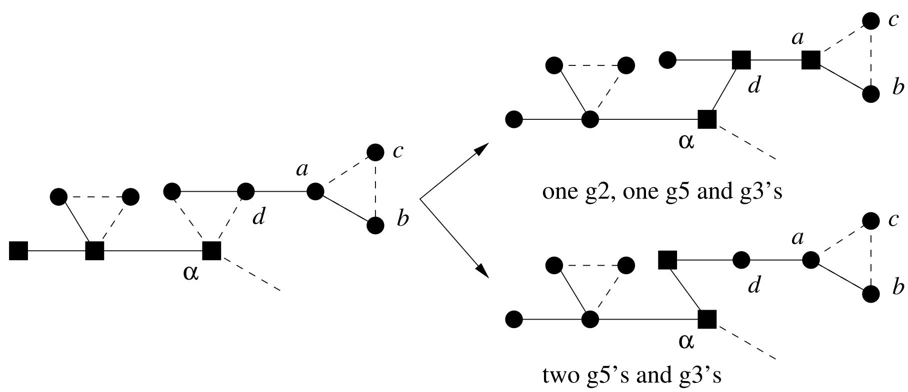


Figure 28: last cases with a g1 in part (a) of Theorem 13, when g is a g3

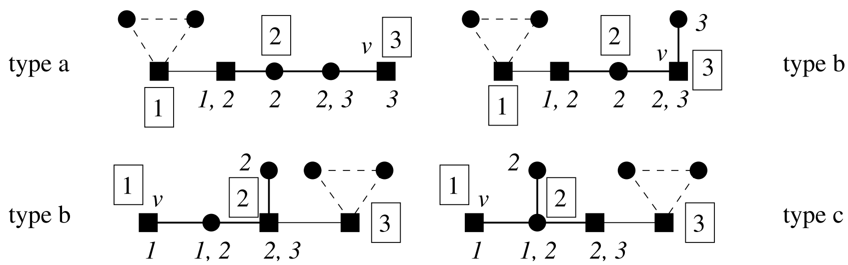


Figure 29: choice of a watcher at a binding vertex of a g4

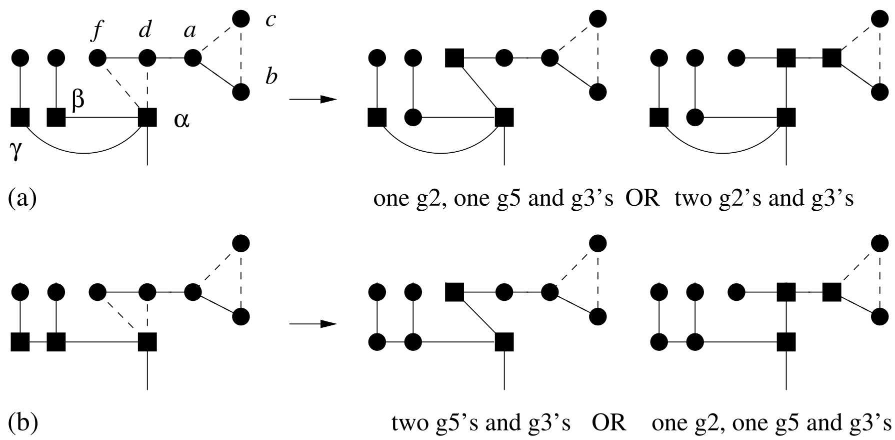


Figure 30: cases with two g2's in part (a) of Theorem 13, when g is a g3

be connected to at least one b. v. of a g3, this means that we can locate one watcher at one b. v. of the g4 in order to possibly cover α. This remark or Lemma 8 or 6 allows us to save one watcher on g when there is one g4 or (at least) one g5 in T', as we did in the left part of Figure 25(a), with β covering α.

Therefore, we have only one case left when g is a g3: when T' contains at least one gadget of order 2. If there is exactly one g2 (case (ii) of Theorem 13), the situation is very close to that of Theorem 12 (see Figure 18 and the left part of Figure 17), the difference being the existence of a g5. So we assume that T' contains two g2's, and g3's (case (i) of Theorem 13). In general, one can still save one watcher on g; the two critical situations are given by Figure 30, in which a single watcher located on g cannot simultaneously cover d, f and β (and γ when γ is connected to α) — cf. Figure 24.

• (3) Assume now that g is of order 5, with binding vertex α ($\alpha \neq d$). The other gadgets in T' are either one g2 and $k-3$ g3's, or one g5 and $k-4$ g3's. We illustrate the cases occurring when in T', α is linked to the g2 and only to this gadget: Figure 31 is for $n = 10$ and uses the only

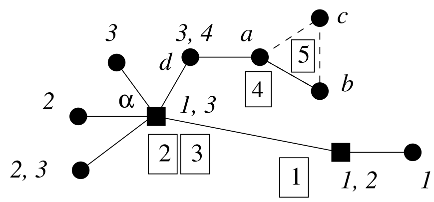


Figure 31: case $n = 10$ in part (a) of Theorem 13, when g is a g5: five watchers are sufficient

representation with one g5 for a tree of order 7, cf. Figure 23(e); Figure 32 is for $n \geq 13$ and is obtained by locating d at all the different vertices, except at the b. v. α, in all the different types for a g5, cf. Figure 20. The other cases, very similar to Figure 32 or to previous studies involving g5's, often use Lemma 9 and are left to the reader.

• (4) The final case of this part (a) is when g is a g4, with its two binding vertices α and x ($\alpha \neq d, x \neq d$) connected to other gadgets, and in T' all gadgets except g are g3's (case (v) of Theorem 13). Denote by T'' the connected component containing x in the forest obtained from T' by removing the edges of g, see Figure 33. Assume that T'' is of order at least 7; then by Lemma 11, x is free in T''. In a minimum watching system for T, we can assume that there is one watcher located at a which covers d, and one watcher located at β which covers α. Then (see Figure 34), either only one more watcher, denoted by 3, is necessary inside g to watch T, and T does not reach the bound $2k$, or T consists of one g2, one g5 and g3's, or of two g5's and g3's. The same argument with α shows that we can assume that each b. v. of g is linked to exactly one g3; then $n = 13$ and Figure 35 gives all the possible cases.

• *Part (b): we assume that we are in the third or fourth case in the proof of Theorem 4*

If we are in the third case, we rename by a, b and d respectively, the vertices x, y and v_{D-1}, and if we are in the fourth case, we rename by a, b and d respectively, the vertices v_{D-1}, v_D and v_{D-2}; in both cases, we remove a and b, obtaining a tree T' of order $3(k-1)+2$. By Remark 5, $w(T') = 2(k-1)+1$ and Theorem 12 may be used: T' can be obtained by choosing one g2 or one g5 and a collection of g3's linked by their binding vertices. Note that the vertex d has degree at least 2 in T'; it belongs to a gadget g with b. v. α.

We first assume that $d \neq \alpha$. Because of the degree of d, the gadget g cannot be of order 2, and if it is of order 3, with vertex set $\{\alpha, d, c\}$, then its edge set is $\{\{d, \alpha\}, \{d, c\}\}$, and $\{a, b, c, d, \alpha\}$ is a g5 of type c or d: T is of the desired form. We are left with the case when g is a g5, in which d has

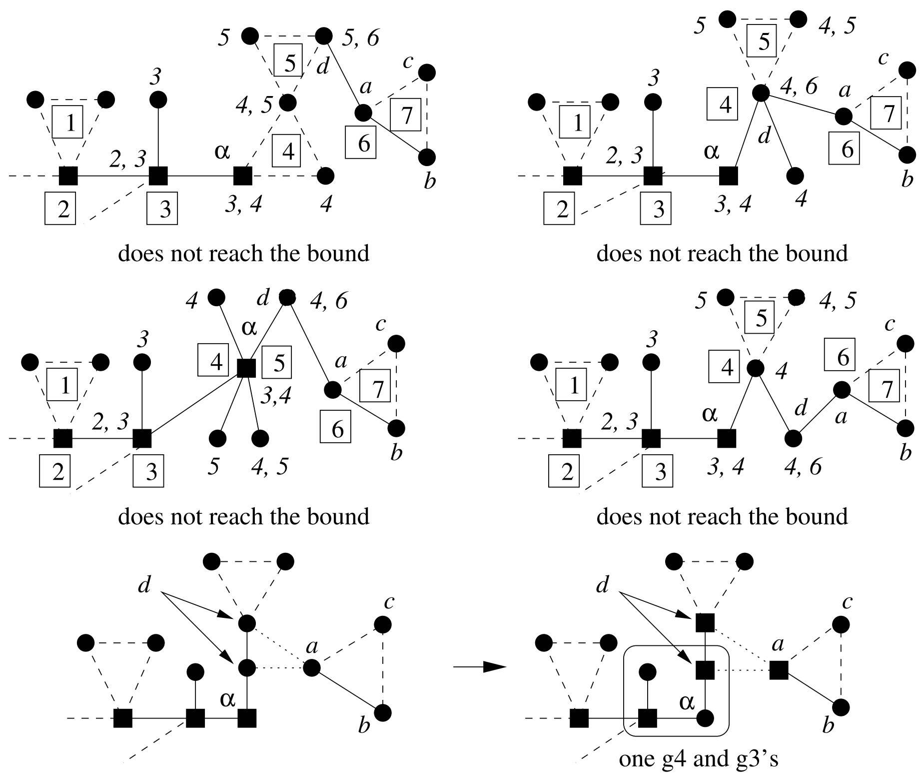


Figure 32: cases when g is a g5 in part (a) of Theorem 13, for $n \geq 13$

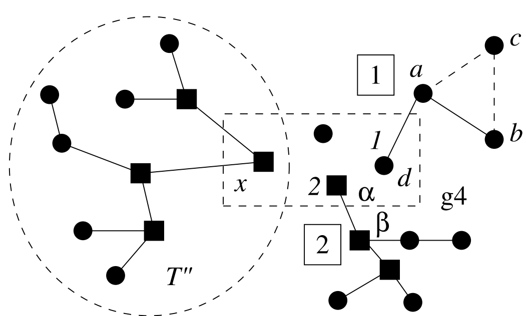


Figure 33: the definition of T'' in part (a) of Theorem 13

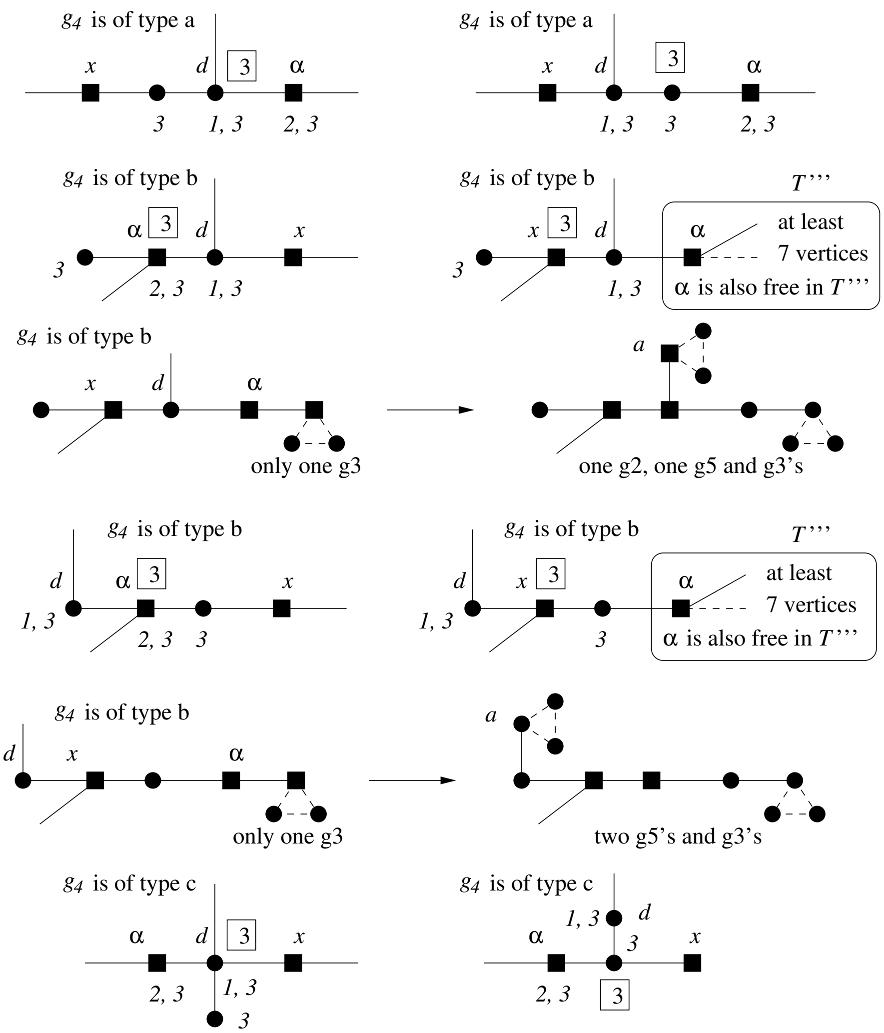


Figure 34: part (a) of Theorem 13: T'' is of order at least 7 and x is free in T''

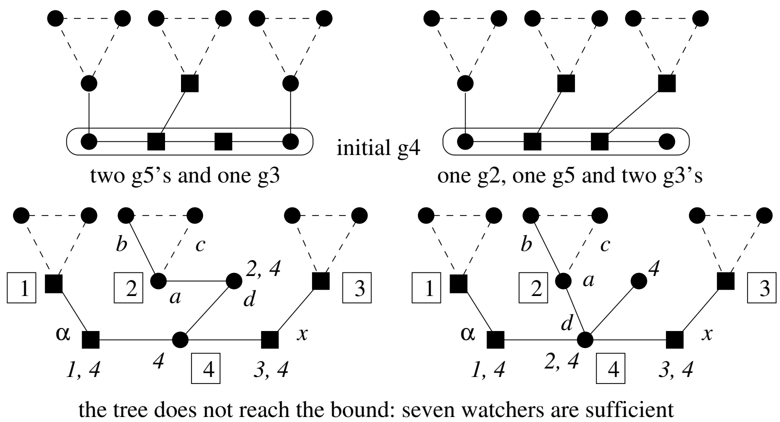


Figure 35: the remaining cases when g is a g4 in part (a) of Theorem 13

degree 2 or more, and the other gadgets in T' are all g3's; this is depicted in Figure 36, where we give the locations of the watchers showing that T does not reach the bound $2k$, or show the b. v.'s of the gadgets involved; note that if $n \geq 13$, then by Lemma 11, α is free in T' deprived of the four ordinary vertices of g.

Finally, if $d = \alpha$, then Figures 37–39 give the different cases, according to the order of g. This completes the proof of Theorem 13. □

4 Graphs G reaching the maximum value of $w(G)$

We first give the following definition.

Definition 14 *A connected graph G is said to be* maximal *if, when we add any edge to G, we obtain a graph G' verifying: $w(G') < w(G)$.*

We denote by $\omega(n)$ the maximum minimum number of watchers needed in a connected graph of order n, i.e.,

$$\omega(n) = \max\{w(G) : G \text{ connected of order } n\}.$$

In the previous section, we have established that $\omega(n) = \lfloor \frac{2n}{3} \rfloor$ for $n \notin \{1, 2, 4\}$, and we have characterized the trees of order n reaching $\omega(n)$. In this section, we want to describe all the maximal connected graphs of order n which reach $\omega(n)$. Using Lemma 1, the graphs of order n which reach $\omega(n)$ are exactly the connected partial graphs of the maximal connected graphs of order n reaching $\omega(n)$.

We recall that K_p denotes the complete graph (or clique) of order p. Again, we divide our study into three cases, $n = 3k$, $n = 3k + 2$ and $n = 3k + 1$.

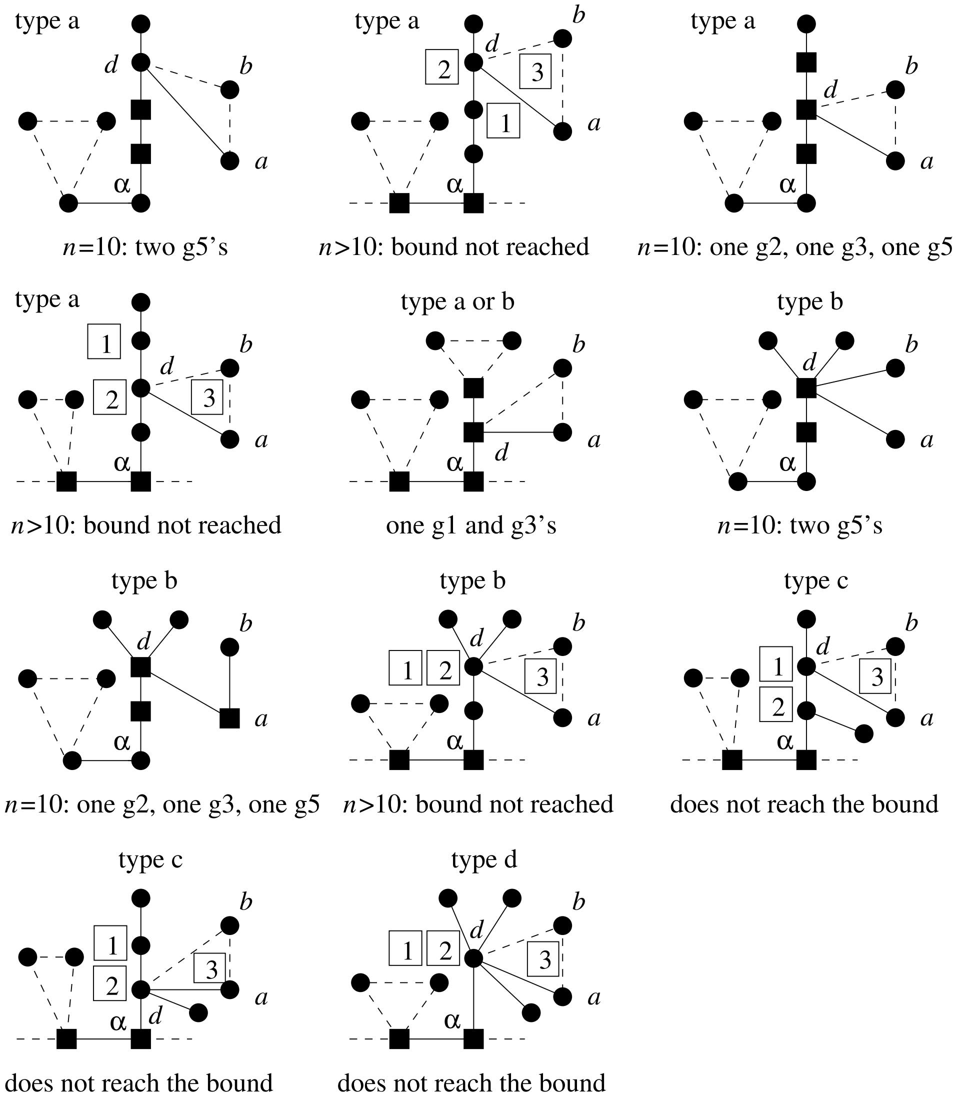


Figure 36: illustration for part (b) of Theorem 13, when g is a g5 and $d \neq \alpha$

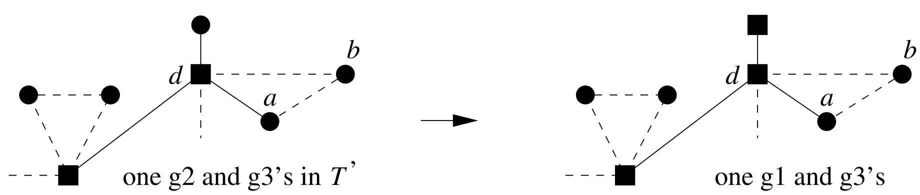


Figure 37: illustration for part (b) of Theorem 13, when $d = \alpha$ and g is a g2

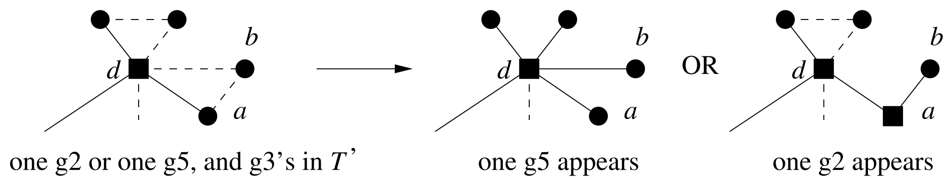


Figure 38: illustration for part (b) of Theorem 13, when $d = \alpha$ and g is a g3

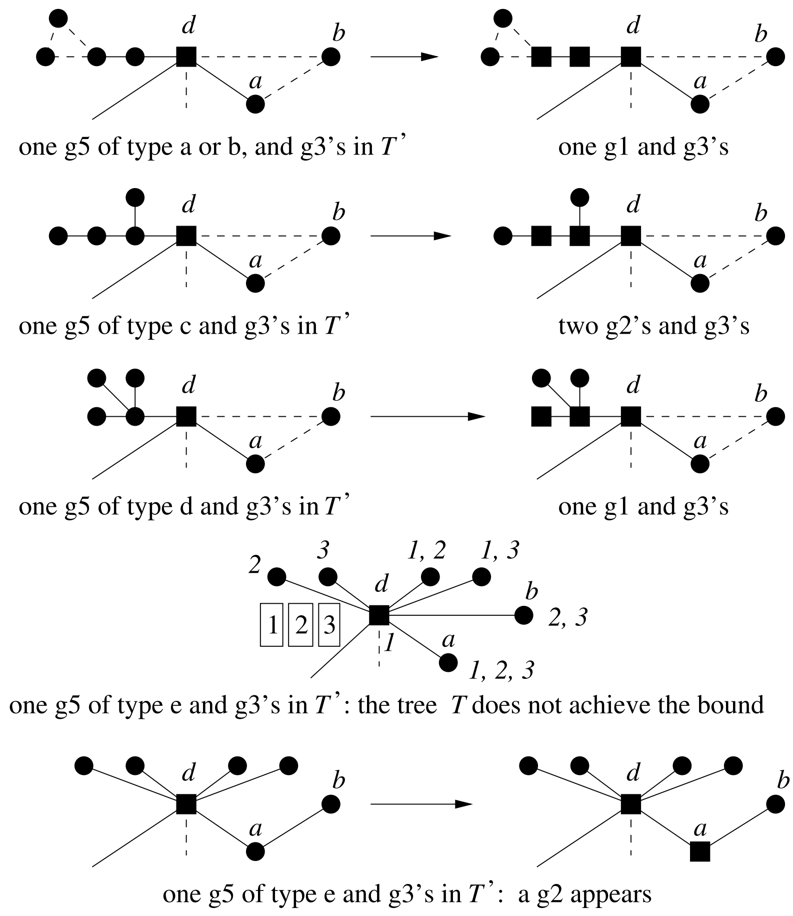


Figure 39: illustration for part (b) of Theorem 13, when $d = \alpha$ and g is a g5

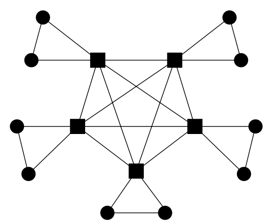


Figure 40: G_{15}, the maximal graph of order 15 reaching the bound

Theorem 15 *Let k be an integer, $k \geq 1$, and G be a maximal graph of order $3k$. We have:*
$w(G) = 2k \Leftrightarrow G$ is obtained by taking a collection of k K_3's, choosing one vertex named a binding vertex *in each K_3, and connecting these k binding vertices by K_k.*

For instance, the graph G_{15} of Figure 40 is the unique maximal graph of order 15 reaching the bound $\omega(15) = 10$.

Proof. The implication from the right to the left is direct. So, given a maximal graph G of order $3k$ satisfying $w(G) = 2k$, we have to prove that G is of the form described in the theorem. Let T be a spanning tree of G. Using Lemma 1 and Theorem 4, we can see that $w(T) = 2k$. By Theorem 7, T is a collection of k gadgets of order 3 connected by their binding vertices. We shall show that in G any edge which is not in T is an edge between two b. v.'s of T, or is the missing edge of a g3; to do this, we assume that there is in G an edge e which is not an edge between two b. v.'s of T, nor the missing edge of a g3. In Figure 41, we consider the four possibilities:

(a) The edge e links an ordinary vertex a of a g3, denoted by g_3, whose b. v. is denoted by β, and the b. v. α of another g3, and the edge $\{\alpha, \beta\}$ exists; then, whatever the type of g_3, we can locate a watcher 3 on g_3 covering a, b and α, and the six vertices are covered and separated by three watchers only.

(b) e links two ordinary vertices of two g3's which are linked by their b. v.'s. Again, the six vertices involved can be watched by three watchers.

In passing, these two cases show how to handle the case $n = 6$, so from now on we assume that $n \geq 9$.

(c) e links an ordinary vertex of a g3, whose b. v. is β, and the b. v. α of another g3, and $\{\alpha, \beta\}$ does not exist. Then α and β are linked to at least one other g3 (possibly the same), because in the spanning tree T, there is a connexion between any two b. v.'s.

(d) This is also true when e links two ordinary vertices of two g3's which are not linked by their b. v.'s.

In each of these two cases, we can see that we are able to locate only one watcher on a g3, so there is a contradiction with the value of $w(G)$.

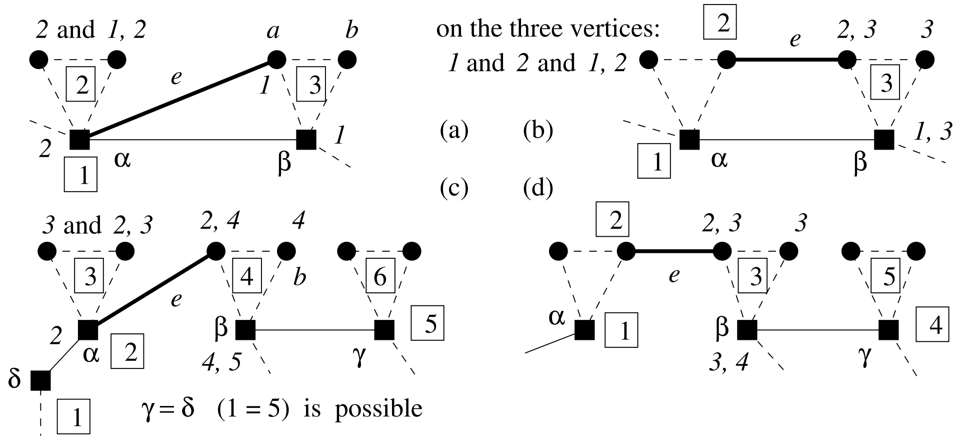


Figure 41: forbidden edges between two g3's in the proof of Theorem 15

Furthermore, if we add to T the missing edge on each g3 and all the missing edges between the b. v.'s of T, the number of needed watchers remains equal to $2k$: we have obtained the unique maximal graph containing T. □

Theorem 16 *(a) Let k be an integer, $k \geq 3$, and G be a maximal graph of order $3k+2$. We have:*
$w(G) = 2k+1 \Leftrightarrow G$ is obtained by taking a collection of k K_3's and one K_2, or $k-1$ K_3's and one K_5, choosing one vertex named a binding vertex in each of these complete graphs, and connecting these binding vertices by K_{k+1} if we have taken a K_2, and by K_k if we have taken a K_5.

(b) If G is a maximal graph of order 8, then we have:
$w(G) = 5 \Leftrightarrow G$ is the graph given by Figure 43, or G is obtained by following the rules given in Case (a), for $k = 2$.

(c) The only maximal graph G of order 5 with $w(G) = 3$ is the clique K_5.

For instance, the graphs G_{17} and G'_{17} of Figure 42 are the two maximal graphs of order 17 reaching the bound $\omega(17) = 11$.

Proof. The implications from the right to the left are direct. So, given a maximal graph G of order $3k+2$ satisfying $w(G) = 2k+1$, we have to prove that G is of the form(s) described in the theorem.

By inequality (1) from the Introduction and Theorem 4, all connected graphs G of order 5 are such that $w(G) = 3$, K_5 is the unique maximal graph of order 5, and Case (c) is true.

The case $n = 8$, which does not fit the general framework either, is rather tedious to check, and is not given here.

We assume from now on that $n \geq 11$. Let T be a spanning tree of G. Using Lemma 1 and Theorem 4, we can see that $w(T) = 2k+1$. From Theorem 12, T can be obtained as one g2 or one g5 plus a collection of g3's,

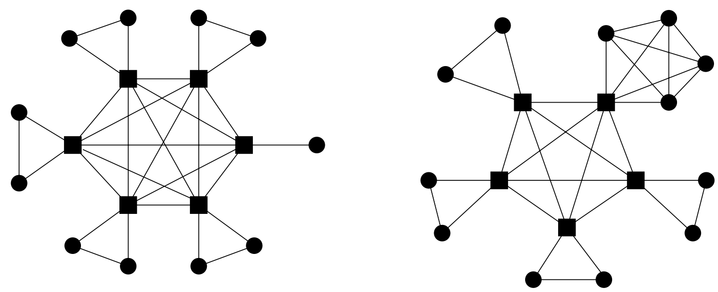


Figure 42: the two maximal graphs of order 17 reaching the bound

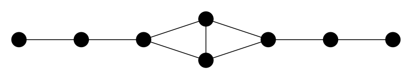


Figure 43: a maximal graph of order 8 reaching the bound

with the gadgets connected by their binding vertices to form a tree. If, among the spanning trees of G, there is one with a g5, we choose this tree; and if, in all the spanning trees, we cannot avoid a g2, then we choose a tree in which the b. v. of the g2 has maximum degree (in the tree).

We shall list pairs of vertices which cannot be adjacent in the maximal graph G: between g3's, between the g5 and a g3, and between the g2 and a g3 (the most delicate case).

• (1) Assume first that there is an edge between two g3's, with at least one of its ends different from a b. v. This case has been treated for Theorem 15, cf. Figure 41. If now β, the b. v. of g_3, is not linked to any b. v. other than α, or if β is linked to the b. v. of a g3 other than α, then we can save one watcher in exactly the same way as on Figure 41. If β is linked to the b. v. γ of the g5, by Lemma 8 we can have a watcher located at γ and covering β, thus still saving one watcher on g_3. So we can assume that β is linked to the b. v. γ of the g2. In cases (b) and (d) of Figure 41, we can save one watcher on the g3 with b. v. α, since α and β play symmetrical parts. In case (c), in all cases, but one, we can still save one watcher on g_3: the critical case (see Figure 44) is when the g2 has no connexion other than β, and moreover the watcher 4, which is used to cover b, cannot be located at β, so that the two vertices of the g2 are not separated, and we cannot save one watcher; in this case however, since the b. v.'s $\alpha, \beta, \gamma, \delta, \ldots$ in T are connected, it is possible to add in T the edge $e = \{\alpha, a\}$ and delete one edge between two b. v.'s, so that the result is a spanning tree of G, in which γ becomes an ordinary vertex in a g3, and a becomes the b. v. of the g2, now connected to two g3's. This means that the spanning tree in the left part of the figure cannot have been chosen, since the b. v. of its g2, γ, does not have

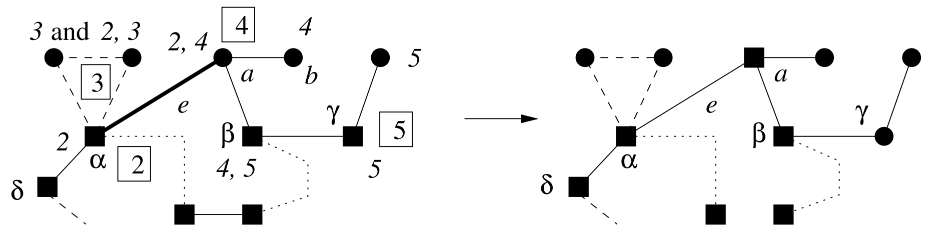


Figure 44: forbidden edges between two g3's: a critical case in part (1) of Theorem 16

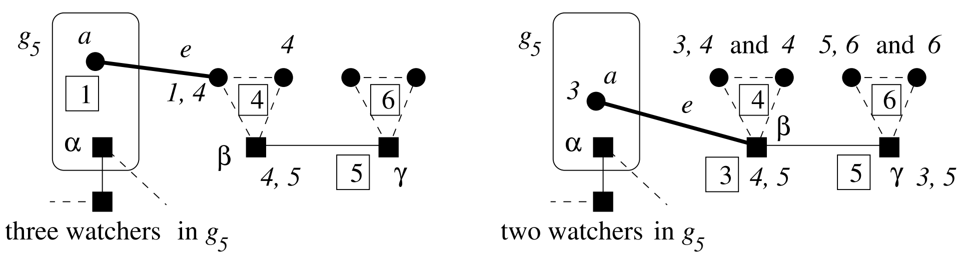


Figure 45: part (2) of Theorem 16: forbidden edges between a g5 and a g3

maximum degree among the spanning trees of G. Case (a) of Figure 41 can be dealt with in the same way, with a critical situation similar to Figure 44, where we can add the edge $\{\alpha, a\}$ and delete the edge $\{\alpha, \beta\}$.

• (2) Assume next that there is one g5, named g_5, in T, and that there is in G an edge e between g_5 and a g3, g_3, with at least one of its ends different from a b. v. Let α and β be the b. v.'s of g_5 and g_3, respectively. If the edge $\{\alpha, \beta\}$ does not exist, then α and β are connected to at least one other g3 (possibly the same), and we can save one watcher, using in particular Lemma 8: see Figure 45, where a can be equal to α in the left part. In the right part, since α is free in the tree T' consisting of the spanning tree T deprived of the four ordinary vertices of g_5 (even if α is linked only to γ, in which case α is covered by the watcher 5), we are left with the problem of taking care of the three vertices x, y, z of g_5 other than a and α, with only two watchers; this can be done using Lemma 9.

So from now on we assume that we have the edge $\{\alpha, \beta\}$ in T. Because $n \geq 11$, α is still free in T', and obviously, if both α and β are still connected to other g3's, the argument above still works. So we assume that only one of α and β is connected to (at least) one (other) g3. We first consider the case when it is β.

Figure 46 depicts the situation, where a can be equal to α in the left part. In this left part, thanks to Lemma 8, the situation is the same as previously (without the edge $\{\alpha, \beta\}$). And if e links a and β (see the right part), then,

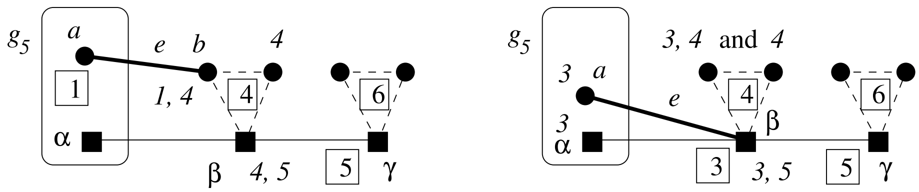


Figure 46: part (2) of Theorem 16: forbidden edges between a g5 and a g3, with $\{\alpha, \beta\}$ in T

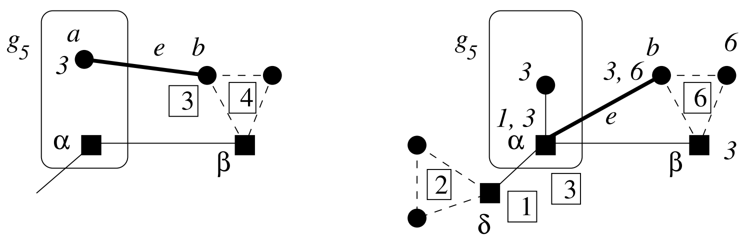


Figure 47: part (2) of Theorem 16: more forbidden edges between a g5 and a g3, with $\{\alpha, \beta\}$ in T

still denoting by x, y and z the vertices in g_5 other than a and α, we can use Lemma 9: two watchers are sufficient to watch $\{x, y, z\}$ and cover a, so that a and α are now separated by a watcher. Thus, one watcher can be saved on g_5. We can now assume that β is linked to no gadget other than g_5.

Then the situation is described by Figure 47, with α free in T', and $b \neq \beta$ or $b = \beta$ in the left part (in the latter case, locate the watcher 3 at β). When a is one extremity of e, we use Lemma 9 and save one watcher on $g5$, so we are left with the case $e = \{\alpha, b\}$ with $b \neq \beta$ (see the right part of the figure), which is solved also using Lemma 9 and saving one watcher on g_3.

• (3) We finally study the case when there is one g2, named g_2, with b. v. α and ordinary vertex α', in the spanning tree T. The situation is now slightly different from the previous cases, because we may, without contradiction, have in G an edge between, for instance, α' and a vertex of a g3, since T may have been originally produced from K_5 in G.

We consider in T a $g3$ named g_3, with b. v. β, and investigate which edge(s) can exist in G between g_2 and g_3.

First, we assume that $\{\alpha, \beta\}$ is not in T. Then in T, α is linked to the b. v. γ of a g3, and β is linked to the b. v. δ of a g3, possibly with $\gamma = \delta$. Using Lemma 6, we locate watchers at γ and δ, and Figure 48(1)(2) shows how to routinely save one watcher on g_3 when in G there is an edge between α or α' and an ordinary vertex of g_3, even if the watchers 3 and 4 coincide. Assume now that it is the edge $\{\alpha', \beta\}$ which is in G. If in T,

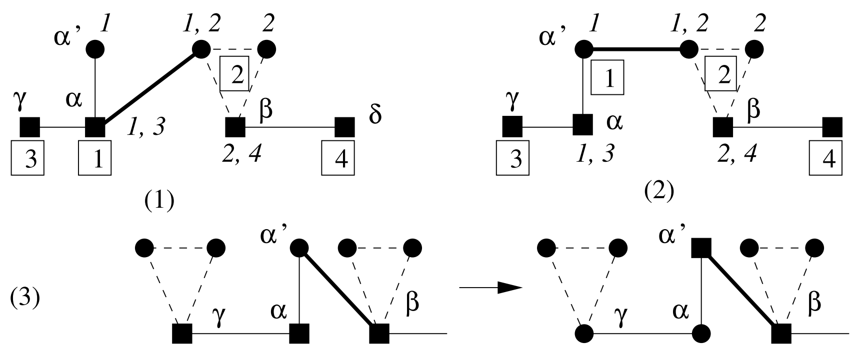


Figure 48: part (3) of Theorem 16: forbidden edges between a g2 and a g3

neither α nor γ is connected to any g3, we are in case (3) of Figure 48 and we consider that there is a g5 of type a or b in T rather than a g2. So either α or γ is connected to a g3, with b. v. ϕ. If $\gamma \neq \delta$, then $\phi = \delta$ is possible, or (if ϕ is not linked to α) $\phi = \beta$; if $\gamma = \delta$, then $\phi = \beta$ is possible (if ϕ is not linked to α). All this is depicted in Figure 49, where it can easily be seen how to save one watcher on g_2 in all cases; we give only the full description of the last case, (e).

So we have just established that if α is not connected to the b. v. of a g3, then there exists no edge between this g3 and g_2 in G. What happens now if α is connected to the b. v. β of a g3, g_3, that is to say if there is the edge $\{\alpha, \beta\}$ in T? If in T, α is still linked to the b. v. γ of a g3 (with $\gamma \neq \beta$) and β is still linked to the b. v. δ of a g3 (with $\gamma \neq \delta$ because there is no cycle in T), we can re-run the argument used in the absence of $\{\alpha, \beta\}$: the first two cases of Figure 48 are exactly the same with or without $\{\alpha, \beta\}$, and in the third case, we have the edges $\{\alpha', \beta\}$ and $\{\alpha, \beta\}$ in G, from which we can still pick a spanning tree with a g5; and, because $\phi \neq \beta$ and $\phi \neq \delta$, Figure 49 reduces to its first case (a), which can be treated similarly. Therefore, in T, either α is not linked to the b. v. of any g3 other than β, or β is not linked to the b. v. of any g3.

If in T, α is linked to β only, then we have seen that no edge exists in G between g_2 and any g3 other than g_3. But edges can exist between g_2 and g_3, and indeed, we can add all the missing edges between these two gadgets, plus the missing edge in each g3, plus all the missing edges between the b. v.'s of T, the number of needed watchers remains equal to $2k + 1$, and we have obtained the only maximal graph containing T, which is of the form described in the theorem; see Figure 50. Note that in some cases, the argument of the choice of a g5 in T can also be used, for instance if g_3 is of type b and there is the edge $\{a, \alpha'\}$.

If β is linked only to α and if β is the only b. v. which is linked only to α, then any g3 other than g_3, with b. v. γ, can be linked to g_2 uniquely

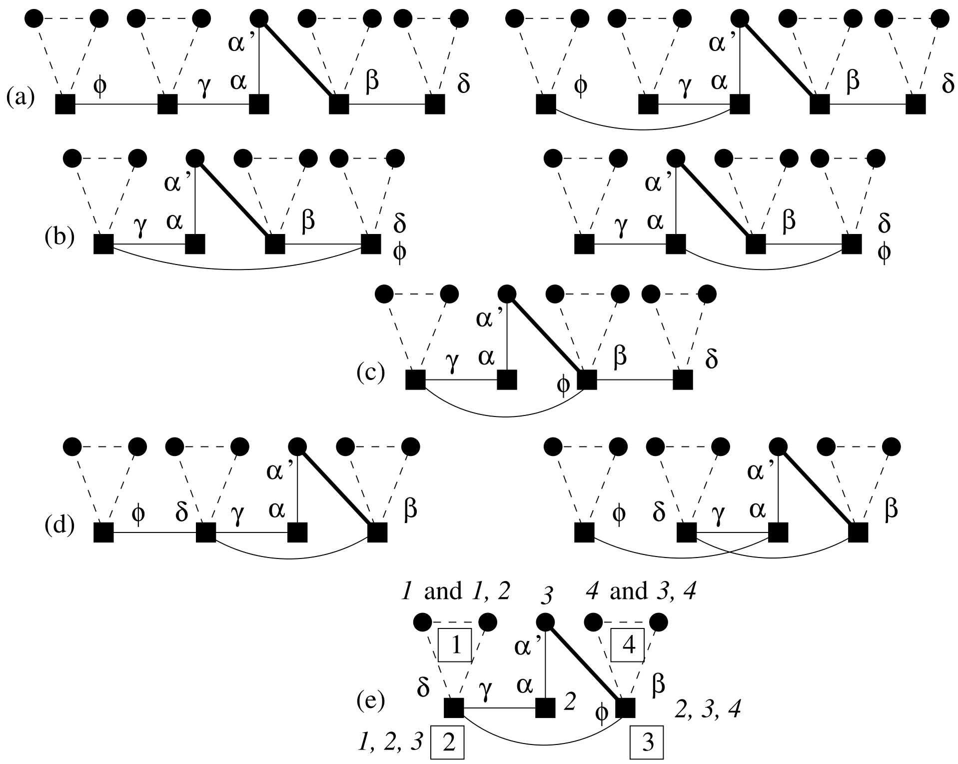


Figure 49: part (3) of Theorem 16, with the edge $\{\alpha, \beta\}$ not in T: (a) $\gamma \neq \delta$, $\phi \neq \delta$, $\phi \neq \beta$; (b) $\gamma \neq \delta$, $\phi = \delta$; (c) $\gamma \neq \delta$, $\phi = \beta$; (d) $\gamma = \delta$, $\phi \neq \beta$; (e) $\gamma = \delta$, $\phi = \beta$.

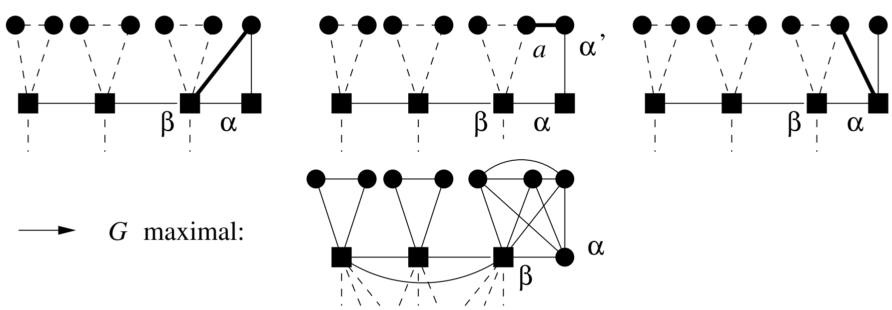


Figure 50: part (3) of Theorem 16, possible edges between a g2 and a g3: actually, edges inside K_5

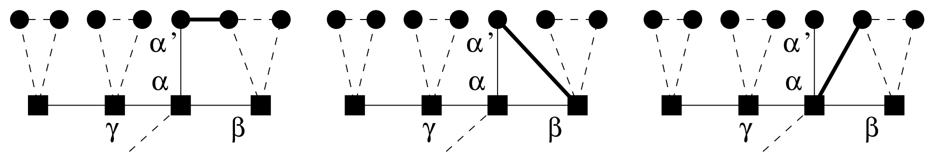


Figure 51: part (3) of Theorem 16: β is the only binding vertex linked only to α

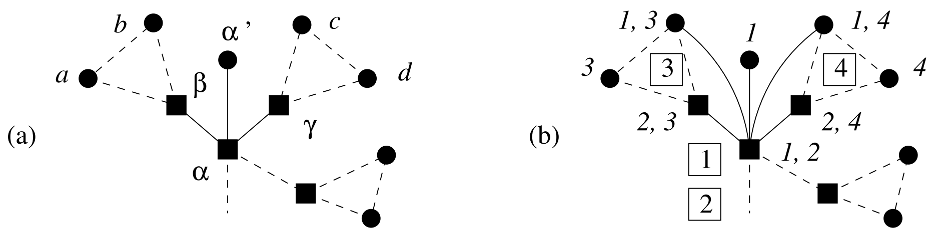


Figure 52: part (3) of Theorem 16: the binding vertices of (at least) two g3's are connected only to α

through the edge $\{\alpha, \gamma\}$, and so in G, the possible edges between the ordinary vertex α' of g_2 and a g3 must affect g_3 only. In the first two cases in Figure 51, a g5 should have been taken when choosing T, or, as in the third case and as in the previous figure, we can add all the edges between g_2 and g_3 and obtain K_5. So we are left with the case when there are two (or more) g3's with b. v.'s linked only to α in T, see Figure 52(a). If in G there is the edge $\{\alpha', a\}$, $\{\alpha', b\}$ or $\{\alpha', \beta\}$, then again a spanning tree with a g5 could have been chosen, and if there is the edge $\{\alpha, a\}$ or $\{\alpha, b\}$ and neither $\{\alpha, c\}$ nor $\{\alpha, d\}$, we can add to T all the edges between g_2 and g_3 in order to obtain K_5 in a maximal graph. So the only possibility not ruled out yet is if there are the edges, say, $\{\alpha, b\}$ and $\{\alpha, c\}$ (more edges in G can only help). Then Figure 52(b) shows how to save (at least) one watcher, by locating two watchers at α.

Now we are in a position to conclude. If in T there are edges between ordinary vertices of different gadgets or between the b. v. of a gadget and an ordinary vertex of another gadget, then another spanning tree should have been chosen, containing a g5 instead of a g2, or containing a g2 with binding vertex of higher degree, or these edges are part of K_5, or we can save watchers.

Furthermore, if we add to T the missing edge on each g3, the missing edges on the possible g5, and all the missing edges between the b. v.'s in T, the number of needed watchers remains equal to $2k+1$: we have obtained the only maximal graph containing T. The proof of Theorem 16 is completed.

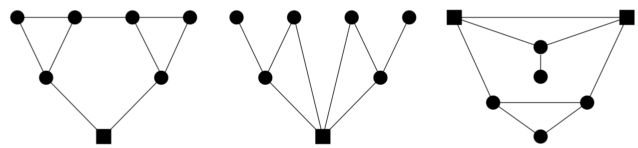


Figure 53: the three new gadgets of order 7

□

The proof of the previous theorem, for $n = 3k+2$, is not very encouraging in view of the case $n = 3k+1$. Indeed, although we have some insight into the situation, we can only conjecture the following result, in which, to describe the graphs, we need three new gadgets of order 7 (which are not trees), with one or two binding vertex(ices), see Figure 53.

Conjecture 17 *Let k be an integer, $k \geq 6$, and G be a maximal graph of order $3k+1$. We have:*
$w(G) = 2k \Leftrightarrow G$ is obtained by:

- (i) *taking two K_2's and $k-1$ K_3's,*
- (ii) *or taking one K_2, one K_5 and $k-2$ K_3's,*
- (iii) *or taking two K_5's and $k-3$ K_3's,*
- (iv) *or taking one K_4 and $k-1$ K_3's,*
- (v) *or taking one g7 and $k-2$ K_3's,*

choosing one vertex named a binding vertex on each of the complete components K_i, except on K_4 for which we choose two binding vertices, taking for the g7 one or two binding vertex(ices) according to its type, and connecting these binding vertices to form a complete graph with them.

The graphs of Figure 54 are graphs of order 19 reaching the bound $\omega(19) = 12$: (a) with one K_2, one K_5 and four K_3's; (b) with one K_4 and five K_3's; (c) with one g7 and four K_3's; according to Conjecture 17, they would be maximal.

For $n = 3k+1$ with $k \leq 5$, there are maximal graphs needing $2k$ watchers which are not of the form described in the conjecture. We give a certified example for $n = 16$ in Figure 55.

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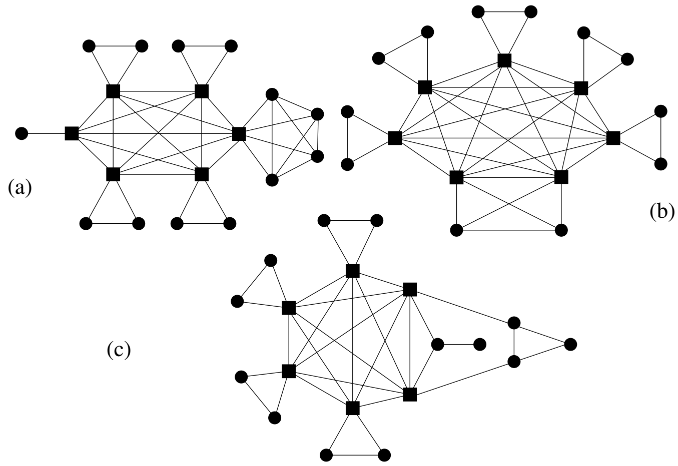

Figure 54: three graphs reaching the bound $\omega(19) = 12$

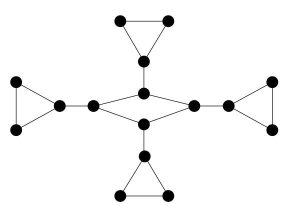

Figure 55: a maximal graph reaching the bound $\omega(16) = 10$

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